Slide 1

Administrivia

- Homework 7 due date moved to Monday. Several problems dropped. (If you had already done them, turn them in for extra credit.)
- Material from textbook section 4.2 (PERT charts and topological sorting) is worth reading, but for reasons of time I won't cover it more in lecture, and you won't be tested on it.

Slide 2

Functions

- Formal definition: $f : S \rightarrow T$ is a subset of $S \times T$, such that for every $s \in S$, there's *exactly one* (s, t) in the subset. Write $f(s) = t$.
- Terminology: S is f's *domain*. T is f's *co-domain* (or *range*).
- Examples:
  - $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $f(x) = x^2$.
  - $g : \mathbb{N} \rightarrow \mathbb{R}$ defined by $g(x) = \sqrt{x}$.
  - $h : P \rightarrow (P \times P)$ (where P is the set of people in the world) defined by $h(x)$ = ((bio?)mother of x, (bio?)father of x).
- Idea easily extends to functions of more than one variable.

Slide 3

Properties of Functions

- For $f : S \to T$, f is *onto* if for every $t \in T$ there's an $s \in S$ with $f(s) = t$. "f covers everything in T."
- For $f : S \to T$, f is *one-to-one* if for every $s, s' \in S$, $f(s) = f(s') \;\; \to \;\; s = s'$. "$f$ maps different things in S to different things in T".
- If f is both one-to-one and onto, call it a *bijection*.

Slide 4

Composition of Functions

- For $f : S \to T$ and $g : T \to U$, can define $g \circ f : ? \to ?$: $(g \circ f)(s) = g(f(s))$.

Function Inverses

Slide 5

- If f is a bijection, can define *inverse* of f, $f^{-1} : T \to S$ such that
  $f^{-1} \circ f$ = identity function on S
  $f \circ f^{-1}$ = identity function on T
- Can we do this if f is not a bijection?

Set Cardinality, Revisited

Slide 6

- We can say that sets S and T have the same cardinality ("same size") if there is a bijection $f : S \to T$ — more formal/precise version of earlier definition, works for both finite and infinite sets.
- If we can define a one-to-one $f : S \to T$, then the cardinality of S is less than or equal to the cardinality of T.
- Recall that we had a "smallest" infinite set $\mathbb{N}$, and a strictly "larger" infinite set $\mathbb{R}$. Are there any bigger sets?
  Yes. Recall that if S is finite with n elements, $\mathscr{P}(S)$ is strictly bigger (2^n elements). True for infinite sets as well — Cantor's theorem.
- Notice that this defines an equivalence relation on sets.

Slide 7

Order of Magnitude of Functions

- By now you've probably heard "this is an $O(N)$ algorithm", etc., many times. Here we'll define it formally.
- First: When we talked about analysis of algorithms (chapter 2), we came up with estimates of "total work" of the algorithm as a function of size of input ("problem size"). Useful and interesting, but a bit fine-grained — what we usually care about is behavior as problem size gets very big.
- So — idea is to come up with an "order of magnitude" for functions, analogous to "order of magnitude" for numbers. If the functions for two algorithms have the same order of magnitude, the functions are in some sense about equally fast/slow.
- Example: If you have two algorithms for processing an image with N pixels, one that takes time proportional to $1000N$ and one that takes time proportional to time N^2, which do you pick? (Does the size of N matter?)

Slide 8

Order of Magnitude of Functions, Continued

- How to determine an order of magnitude for functions?

  If we look at graphs of functions, we might notice that we can classify them into groups based on their "shape".

  For nondecreasing functions, we also notice that some shapes "grow" faster than others.

  (Compare x^2, $10x^2$, x^3, etc.)
- Idea is that we want functions that have the same shape to have the same order of magnitude.

Order of Magnitude of Functions, Continued

- Formal definition:

  Write $f = \Theta(g)$ to mean that f and g have the same order magnitude.

  Define to be true iff there are positive constants n_0, c_1, c_2 such that for all $x \geq n_0$

  $$c_1 g(x) \leq f(x) \leq c_2 g(x)$$

  In other words, these functions are roughly proportional to each other.

- Try an example: $f(x) = x - 10$, $g(x) = 3x + 2$.
- Of course this is incredibly tedious, so people have come up with (and proved) general rules for polynomials, other common functions.

Minute Essay

- For each of the following functions, is it one-to-one? onto?
  - $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = |x|$.
  - $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined by $f(x) = \sqrt{x}$. ($\mathbb{R}^+$ is the positive real numbers.)

Minute Essay Answer

- $f : \mathbb{R} \rightarrow \mathbb{R}$ is neither one-to-one ($|1| = |-1|$, for example) nor onto (there's no x such that $|x| = -1$, for example).
- $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined by $f(x) = \sqrt{x}$ is both one-to-one and onto.

Slide 11