Slide 1

Administrivia

- Homework 9 to be on Web later today; due next Friday.
- In the reading for today (4.2 and 4.4), it's okay to skim/skip the material on PERT charts (p. 272), permutation functions (pp. 300–302), and how many functions (pp. 302–306).

Slide 2

Recap — Binary Relations

- Idea of a binary relation is to express relationship between pairs of elements of a set.
- Several properties of interest — reflexivity, symmetry, transitivity, etc. — which allow us to define some special cases:
  - Partial orderings (generalization of "less than or equal to") — useful in working out how we could put things "in order", e.g., a set of tasks with (some) ordering dependencies.
  - Equivalence relations (generalization of "equals").

Slide 3

Topological Sorting

- Idea here is to take a partial ordering and find a way to extend it to a "total" ordering (i.e., add pairs so that for every x and y either $x \; \rho \; y$ or $y \; \rho \; x$. How is this useful? e.g., find a way to "schedule" interdependent tasks.
- Notice that there could be more than one way to do this for a given partial ordering.
- How to do this? Next slide . . .

Slide 4

Topological Sorting, Continued

- Algorithm for finding *a* way to extend a partial ordering — "topological sort":
- Start with set S and partial ordering ρ on S. Idea is to turn S into a sequence $x_1, x_2, \ldots$ such that $(x_i \; \rho \; x_j) \;\rightarrow\; (i \leq j)$.
- The algorithm might look like this in pseudocode:

  while (S not empty)
  pick a minimal element x in S
  make it the next element of the sequence and remove it from S
  end while

  ("Minimal" here means an element such that aren't any that are smaller.)
- Does this work? i.e., does it produce an ordering that extends ρ? True if we can be sure that for x and y with $x \; \rho \; y$ x is picked before y.

Slide 5

Functions

- Formal definition: $f : S \rightarrow T$ is a subset of $S \times T$, such that for every $s \in S$, there's *exactly one* (s, t) in the subset. Write $f(s) = t$.
- Terminology: S is f's *domain*. T is f's *co-domain* (or *range*).
- Examples:
  - $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $f(x) = x^2$.
  - $g : \mathbb{N} \rightarrow \mathbb{R}$ defined by $g(x) = \sqrt{x}$.
  - $h : P \rightarrow (P \times P)$ (where P is the set of people in the world) defined by $h(x)$ = ((bio?)mother of x, (bio?)father of x).
- Idea easily extends to functions of more than one variable.

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Properties of Functions

- For $f : S \rightarrow T$, f is *onto* if for every $t \in T$ there's an $s \in S$ with $f(s) = t$. "f covers everything in T."
- For $f : S \rightarrow T$, f is *one-to-one* if for every $s, s' \in S$, $f(s) = f(s') \rightarrow s = s'$. "$f$ maps different things in S to different things in T".
- If f is both one-to-one and onto, call it a *bijection*.

Composition of Functions

- For $f : S \to T$ and $g : T \to U$, can define $g \circ f : ? \to ?$:
  $(g \circ f)(s) = g(f(s))$.

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Function Inverses

- If f is a bijection, can define *inverse* of f, $f^{-1} : T \to S$ such that
  $f^{-1} \circ f$ = identity function on S
  $f \circ f^{-1}$ = identity function on T
- Can we do this if f is not a bijection?

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Slide 9

Set Cardinality, Revisited

- We can say that sets S and T have the same cardinality ("same size") if there is a bijection $f : S \to T$ — more formal/precise version of earlier definition, works for both finite and infinite sets.
- If we can define a one-to-one $f : S \to T$, then the cardinality of S is less than or equal to the cardinality of T.
- Recall that we had a "smallest" infinite set $\mathbb{N}$, and a strictly "larger" infinite set $\mathbb{R}$. Are there any bigger sets?

  Yes. Recall that if S is finite with n elements, $\mathscr{P}(S)$ is strictly bigger (2^n elements). True for infinite sets as well — Cantor's theorem.
- Notice that this defines an equivalence relation on sets.

Slide 10

Order of Magnitude of Functions

- By now you've probably heard "this is an $O(N)$ algorithm", etc., many times. Here we'll define it formally.
- First: When we talked about analysis of algorithms (chapter 2), we came up with estimates of "total work" of the algorithm as a function of size of input ("problem size"). Useful and interesting, but a bit fine-grained — what we usually care about is behavior as problem size gets very big.
- So — idea is to come up with an "order of magnitude" for functions, analogous to "order of magnitude" for numbers. If the functions for two algorithms have the same order of magnitude, the functions are in some sense about equally fast/slow.
- Example: If you have two algorithms for processing an image with N pixels, one that takes time proportional to $1000N$ and one that takes time proportional to time N^2, which do you pick? (Does the size of N matter?)

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Order of Magnitude of Functions, Continued

- How to determine an order of magnitude for functions?

  If we look at graphs of functions, we might notice that we can classify them into groups based on their "shape".

  For nondecreasing functions, we also notice that some shapes "grow" faster than others.

  (Compare x^2, $10x^2$, x^3, etc.)

- Idea is that we want functions that have the same shape to have the same order of magnitude.

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Order of Magnitude of Functions, Continued

- Formal definition:

  Write $f = \Theta(g)$ to mean that f and g have the same order magnitude.

  Define to be true iff there are positive constants n_0, c_1, c_2 such that for all $x \geq n_0$

  $$c_1 g(x) \leq f(x) \leq c_2 g(x)$$

  In other words, these functions are roughly proportional to each other.

- Can guess values c_1, c_2 and more or less show that they work by plotting resulting curves — but to really show that the definition holds, must do algebra to show. Example next time?

- Of course this is incredibly tedious, so people have come up with (and proved) general rules for polynomials, other common functions.

Minute Essay

- For each of the following functions, is it one-to-one? onto?
  - $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = |x|$.
  - $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined by $f(x) = \sqrt{x}$. ($\mathbb{R}^+$ is the positive real numbers.)

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Minute Essay Answer

- $f : \mathbb{R} \rightarrow \mathbb{R}$ is neither one-to-one ($|1| = |-1|$, for example) nor onto (there's no x such that $|x| = -1$, for example).
- $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined by $f(x) = \sqrt{x}$ is both one-to-one and onto.

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