Slide 1

Administrivia

- Reminder: Homework 1 due Tuesday (5pm).
- Tentative quiz dates scheduled. First will be next Thursday. Ten minutes, end of class, open book/notes.
- (E-mail about TA office hours.)

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Recap — Propositional Logic Proofs

- Idea is to construct detailed formal proof ("proof sequence") capturing "valid argument" that one thing logically follows from others.

  Problems sometimes cast in terms of hypotheses and conclusion, sometimes as "prove that $P \wedge Q \rightarrow R$ is a tautology". Same thing — "deduction method."
- Proof sequence can be thought of as sequence of valid moves in an elaborate game. Typically guided by some deeper understanding of why conclusion follows from hypotheses, but — this is a formal system, and we're not allowed to make up new moves, however plausible-seeming, unless we can prove (with a proof sequence) that the new move is valid.

Why Predicate Logic?

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- Propositional logic captures some of what we need to talk about things logically, but not everything.
- Example from classical logic:

  "All humans are mortal. Socrates is human. Therefore Socrates is mortal."

  No way to express this in propositional logic.
- What we want to add is some way to express the idea of something being true "for all x" or "for at least one x".

Predicates

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- Define "predicate" — boolean-valued function of one or more variables.
- Examples with integer variables:

  $P(x) = (x > 0)$

  $Q(x, y) = (x < y)$
- Examples with people variables:

  $P(x)$ means "x is a student in CSCI 1323".

  $Q(x, y)$ means "x is taller than y".

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Quantifiers

- Universal quantification: $(\forall x)P(x)$ means "for all x, $P(x)$ is true."
- Existential quantification: $(\exists x)P(x)$ means "there exists an x such that $P(x)$ is true."
- How to decide whether such a statement is true? For propositional-logic connectives, we could write down a truth table for different values of the formulas being connected. That won't work here. (Why?)
- Instead, notion of a "domain of interpretation" — (non-empty) range of values for the variable, definition of predicate(s).

  $(\forall x)P(x)$ means — ?

  $(\exists x)P(x)$ means — ?

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A Few More Definitions

- Define "variables" (usually write them x, y, etc.) and "constants" (usually write them a, b, etc.) — elements of domain of interpretation.
- "Free variables" are those not within scope of a quantifier — e.g., x but not y in $(\forall y)P(x, y)$.
- Notice that we can change the variable in a quantification — it's a "dummy variable" – as long as we don't duplicate another variable.
- As in propositional logic, can define notion of well-formed formula (wff) — "sensible" combination of predicates, quantifiers, connectives from propositional logic, and parentheses.
- How to express "All men are mortal", etc?

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Interpretations

- Expressions involving predicates are true/false depending on “interpretation” (analogous to assigning values to statements in propositional logic):
  - Domain of the interpretation (must not be empty).
  - Assignment of a property of objects in the domain to each predicate.
  - Assignment of a particular object to each constant symbol.
- Given an interpretation and an expression, we can (usually) compute a value for it. (What if there’s at least one free variable?)

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Interpretations — Example

- Suppose the domain is the integers, $Q(x)$ means “x has an integer square root”, and $c = 0$.
- What is the “truth value” of the following?
  - $Q(4)$
  - $Q(2)$
  - $(\forall x)Q(x)$
  - $(\exists x)Q(x)$
  - $Q(4) \ \vee \ Q(2)$
  - $Q(c)$
  - $Q(x)$

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English to Formulas

- Given people as a domain and predicates
  - $C(x)$ meaning “x is a CS student”
  - $D(x)$ meaning “x must pass CSCI 1323 to graduate”
  - $B(x)$ meaning “x is a business major”
  - $M(x)$ meaning “x likes math”
- Translate (letting “some” mean “at least one”):
  - “‘All CS majors must pass CSCI 1323 to graduate.”
  - “Some CS majors are business majors.”
  - “Some CS majors like math.”
  - “Not all CS majors like math.”

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Propositional Logic Versus Predicate Logic

- In propositional logic:
  - Wffs are true or false, depending on assignment of truth values to statement letters.
  - If a wff is true for all such assignments, “tautology” — always true.
  - Can show this by checking all cases (truth table).
- In predicate logic:
  - Wffs are true or false (or neither, if they have free variables), depending on “interpretation” (domain plus meanings for predicates and constants).
  - If a wff is true for all such interpretations, “valid” — always true.
  - *Cannot* show this by checking all cases.

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Valid Arguments, Revisited

- As with propositional logic, we want to know when we can say that a conclusion "logically follows" from a set of hypotheses — i.e., no matter what interpretation we choose, if the hypotheses are true so is the conclusion.
- What we have in our "bag of tricks":
  - All propositional-logic rules.
  - New rules for manipulating quantifiers.

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Universal Instantiation

- Rule for removing $\forall$. (Why do we want to do this?)
- If we have $(\forall x)P(x)$
  we can write $P(t)$
  provided t doesn't already exist "bound" in $P(x)$.
- "If $P(x)$ for all x, then $P(t)$ for a particular t".

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Existential Instantiation

- Rule for removing $\exists$. (Why do we want to do this?)
- If we have $(\exists x)P(x)$

  we can write $P(t)$

  provided t has not been previously used in the proof.
- "If there is some x for which $P(x)$, we can give it a name — t, for example."

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Universal Generalization

- Rule for introducing $\forall$. (Why do we want to do this?)
- If we have $P(x)$

  we can write $(\forall x)P(x)$

  provided x is "arbitrary" — not a free variable in a hypothesis, not a variable we got from ei, not a free variable in a formula we got from ei. (For last part, consider last part of Example 28.)

  (Yes, this is tricky to understand/apply.)
- "If we know $P(x)$ for arbitrary x, then $P(x)$ for all x."

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Existential Generalization

- Rule for introducing $\exists$. (Why do we want to do this?)
- If we have $P(y)$ or $P(a)$
  we can write $(\exists x)P(x)$
  provided x doesn't appear in $P(a)$.
- "If we have some particular z for which $P(z)$, then there exists an x such that $P(x)$."

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Examples

- Show that

$$(\forall x)P(x) \ \wedge \ (\exists x)Q(x) \ \rightarrow \ (\exists x)(P(x) \ \wedge \ Q(x))$$

- Show that

$$(\forall x)(\forall y)Q(x, y) \ \rightarrow \ (\forall y)(\forall x)Q(x, y)$$

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Minute Essay

- Consider formulas $Q(a)$, $Q(b)$, $(\forall x)Q(x)$. Tell me whether each is true or false for the following interpretations.
- Interpretation 1: domain of interpretation is the integers, $a = 1$, $b = 2$, and $Q(x)$ means "$2x$ is an even integer".
- Interpretation 2: domain of interpretation is the rational numbers, $a = 1/2$, $b = 1$, and $Q(x)$ means "$2x$ is an even integer".

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Minute Essay Answer

- Interpretation 1: domain of interpretation is the integers, $a = 1$, $b = 2$, and $Q(x)$ means "$2x$ is an even integer".

  $Q(a)$ true, $Q(b)$ true, $(\forall x)Q(x)$ true.
- Interpretation 2: domain of interpretation is the rational numbers, $a = 1/2$, $b = 1$, and $Q(x)$ means "$2x$ is an even integer".

  $Q(a)$ false, $Q(b)$ true, $(\forall x)Q(x)$ false.