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Administrivia

- Reminder: Homework 1 due today at 5pm.
- Reminder: Quiz 1 Thursday. Question(s) likely to be about propositional logic. Open book/notes.
- Homework 2 on the Web; due in a week.

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Predicate Logic — Review/Recap

- Propositional logic is enough for some things but not for others — “Socrates is mortal” example.
- Predicate logic gives us more to work with — specifically, adds notions of domain and quantifiers (“for all” and “there exists”).
- Here too we have a notion of “valid argument”, in which we can use all our propositional-logic deduction rules, plus four new ones for adding/removing quantifiers.

  Notice that all of these rules apply to whole formulas only, not to parts of formulas.

Universal Instantiation

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- Rule for removing $\forall$. (Why do we want to do this?)
- If we have $(\forall x)P(x)$

  we can write $P(t)$

  provided t doesn't already exist "bound" in $P(x)$.
- "If $P(x)$ for all x, then $P(t)$ for a particular t".

Universal Instantiation, Continued

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- Why the restriction (can't replace the quantified variable with one that's "bound")? Without it . . .
- Suppose the domain is the integers and $P(x, y)$ means $x < y$. If we have

$$(\forall x)((\exists y)P(x, y))$$

  we could conclude

$$(\exists y)P(y, y)$$

  which we don't want.

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Existential Instantiation

- Rule for removing $\exists$. (Why do we want to do this?)
- If we have $(\exists x)P(x)$

  we can write $P(t)$

  provided t has not been previously used in the proof.
- "If there is some x for which $P(x)$, we can give it a name — t, for example."

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Existential Instantiation, Continued

- Why the restriction (variable must not have been previously used)? Without it . . .
- Suppose the domain is the integers again, and $P(x)$ means $x > 0$ and $Q(x)$ means $x < 0$. If we have

$$(\exists x)P(x)$$

$$(\exists x)Q(x)$$

  we could conclude

$$P(a)$$

$$Q(a)$$

  which we don't want.

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Universal Generalization

- Rule for introducing $\forall$. (Why do we want to do this?)
- If we have $P(x)$

  we can write $(\forall x)P(x)$

  provided x is "arbitrary" — not a free variable in a hypothesis, not a variable we got from ei, not a free variable in a formula we got from ei. (For last part, consider last part of Example 28.)

  (Yes, this is tricky to understand/apply.)
- "If we know $P(x)$ for arbitrary x, then $P(x)$ for all x."

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Universal Generalization, Continued

- Why the restriction (variable must be "arbitrary")? Without it . . .
- Any time we have

$$(\exists x)P(x)$$

$$P(a)$$

  we could conclude that

$$(\forall x)P(x)$$

  which we don't want.

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Existential Generalization

- Rule for introducing $\exists$. (Why do we want to do this?)
- If we have $P(y)$ or $P(a)$
  we can write $(\exists x)P(x)$
  provided x doesn't appear in $P(a)$.
- "If we have some particular z for which $P(z)$, then there exists an x such that $P(x)$."

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Existential Generalization, Continued

- Why the restriction (variable must not appear in formula being generalized)? Without it . . .
- Suppose the domain is the integers and $P(x, y)$ means $x < y$. If we have

$$P(x, y)$$

we could conclude

$$(\exists y)P(y, y)$$

which we don't want.

Examples

- Show that

$$(\forall x)P(x) \ \wedge \ (\exists x)Q(x) \ \rightarrow \ (\exists x)(P(x) \ \wedge \ Q(x))$$

- Show that

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$$(\forall x)(\forall y)Q(x,y) \ \rightarrow \ (\forall y)(\forall x)Q(x,y)$$

Predicate Logic, Recap / What Next?

- Now we have a set of derivation rules for predicate logic (we'll add a few more for convenience later).
- As with propositional logic, we could show that these rules are "sound" (if we can prove something, it's true/valid) and "complete" (if something is true/valid, we can prove it).

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Temporary Hypotheses

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- In propositional logic, we allowed proving a conclusion of the form $P \rightarrow Q$ by adding P to the list of hypotheses and proving Q.
- Along the same lines, we allow "temporary hypotheses":

  Suppose as part of a proof we want to show that $R \rightarrow S$ follows from the hypotheses. If $R \rightarrow S$ is the conclusion, deduction method works. What if it's not? Then we can't just add R to the list of hypotheses. What to do?
- One solution would be (in mathspeak) a lemma ("branch" or side proof).

Temporary Hypotheses, Continued

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- Another solution is basically an inline lemma:
  - Introduce "temporary hypothesis" T.
  - Derive some more steps from earlier results and T, ending with S.
  - Conclude that $T \rightarrow S$.

  Note that the formulas we derive from earlier steps and T might depend on T, so — indent to make it clear that they're not part of the main proof.
- Example — section 1.4 problem 22.

One More Rule, a Conclusion

- One more rule — negation (example 32 p. 56).
- A conclusion — the goal of formal logic is to make arguments as meaningless as possible (!) — i.e., abstract out everything that doesn't matter, and apply formal mathematical rules to what's left.

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Predicate Logic — Proof Sequence Sketch

- Start by removing quantifiers (with ei, ui rules) — usually remove existential quantifiers first, then universal.
- Apply rules from propositional logic to get unquantified result.
- Use eg, ug rules to put quantifiers back in.

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Minute Essay

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- What (if anything!) did you find interesting about Homework 1?
- What (if anything!) did you find difficult about Homework 1?