Slide 1

Administrivia

- None?

Slide 2

Functions

- Definition: $f : S \to T$ is a subset of $S \times T$, such that for every $s \in S$, there's *exactly one* (s, t) in the subset. Write $f(s) = t$.
- Terminology: S is f's *domain*. T is f's *co-domain* (or *range*).
- Examples:
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^2$.
  - $g : \mathbb{N} \to \mathbb{R}$ defined by $g(x) = \sqrt{x}$.
  - $h : P \to (P \times P)$ (where P is the set of people in the world) defined by $h(x)$ = ((bio?)mother of x, (bio?)father of x).
- Idea easily extends to functions of more than one variable.

Slide 3

Properties of Functions

- For $f : S \to T$, f is *onto* if for every $t \in T$ there's an $s \in S$ with $f(s) = t$. "f covers everything in T."

  Examples?

- For $f : S \to T$, f is *one-to-one* if for every $s, s' \in S$, $f(s) = f(s') \;\to\; s = s'$. "$f$ maps different things in S to different things in T".

  Examples?

- If f is both one-to-one and onto, call it a *bijection*.

Slide 4

Composition of Functions

- For $f : S \to T$ and $g : T \to U$, can define $g \circ f :? \to ?$: $(g \circ f)(s) = g(f(s))$.
- Examples?

Slide 5

Function Inverses

- If f is a bijection, can define *inverse* of f, $f^{-1} : T \to S$ such that
  $f \circ f^{-1}$ = identity function on S
  $f^{-1} \circ f$ = identity function on T
- Can we do this if f is not a bijection?

Slide 6

Set Cardinality, Revisited

- We can say that sets S and T have the same cardinality ("same size") if there is a bijection $f : S \to T$ — more formal/precise version of earlier definition, works for both finite and infinite sets.
- If we can define a one-to-one $f : S \to T$, then the cardinality of S is less than or equal to the cardinality of T.
- Recall that we had a "smallest" infinite set $\mathbb{N}$, and a strictly "larger" infinite set $\mathbb{R}$. Are there any bigger sets?
  Yes. Recall that if S is finite with n elements, $\mathscr{P}(S)$ is strictly bigger (2^n elements). True for infinite sets as well — Cantor's theorem.
- Notice that this defines an equivalence relation on sets.

Minute Essay

- For each of the following functions, is it one-to-one? onto?
  - $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$.
  - $f : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $f(x) = \sqrt{x}$. ($\mathbb{R}^+ +$ is the positive real numbers.)

Slide 7