Slide 1

Administrivia

- Reminder: Homework 7 due Thursday.
- (Review minute essay from last time.)

Slide 2

Binary Relations

- Formal definition: A binary relation ρ on a set S is a subset of $S \times S$. Usually this set is defined by some property of interest. For $a, b \in S$, we write $a \ \rho \ b$ iff (if and only if) (a, b) is in this subset.
- Examples:
  - S is people in the world; $x \ \rho \ y$ iff x and y are siblings.
  - S is integers; $x \ \rho \ y$ iff $x < y$.
  - S is integers; $x \ \rho \ y$ iff y is a multiple of x.
  - S is sets of integers; $x \ \rho \ y$ iff $x \subseteq y$.
- Notice that for a given relation ρ and element x, there can be any number (including zero) of y's such that $x \ \rho \ y$ and any number (including zero) of y's such that $y \ \rho \ x$.

Slide 3

Properties of Binary Relations

- ρ is *reflexive* if $x \ \rho \ x$ for all $x \in S$.
- ρ is *symmetric* if $(x \ \rho \ y) \ \rightarrow \ (y \ \rho \ x)$ for all $x, y \in S$.
- ρ is *transitive* if $(x \ \rho \ y) \ \wedge \ (y \ \rho \ z) \ \rightarrow \ (x \ \rho \ z)$ for all $x, y, z \in S$.
- ρ is *antisymmetric* if $(x \ \rho \ y) \ \wedge \ (y \ \rho \ x) \ \rightarrow \ (x = y)$ for all $x, y \in S$.
- Can combine these in interesting ways . . .

Slide 4

Partial Ordering

- Idea: Generalize idea of "ordering" to include relations where not all pairs of elements can be ordered.
- Definition: ρ is a partial ordering if it's reflexive, antisymmetric, and transitive.
- Examples: $\leq$ on integers or reals, $\subseteq$ on sets.

Slide 5

Equivalence Relation

- Idea: Generalize idea of "equals" to include relations where pairs of elements are equivalent but not identical.
- Definition: ρ is an equivalence relation if it's reflexive, symmetric, and transitive.
- Examples: $=$ on integers or reals, $(x \bmod n) = (y \bmod n)$ for some n.
- Related terms/ideas:
  - Equivalence classes.
  - Partition of a set.

Slide 6

Uses of Partial Orderings

- One thing a partial ordering (reflexive, symmetric, transitive relation — think "generalized $\leq$") can express — ordering constraints among tasks.
- We'll look at one application — topological sorting. PERT charts discussed in book.

Slide 7

Topological Sorting

- Idea here is to take a partial ordering and find a way to extend it to a "total" ordering (i.e., add pairs so that for every x and y either $x \; \rho \; y$ or $y \; \rho \; x$. How is this useful? e.g., find a way to "schedule" interdependent tasks.
- Notice that there could be more than one way to do this for a given partial ordering.
- How to do this? Next slide . . .

Slide 8

Topological Sorting, Continued

- Algorithm for finding *a* way to extend a partial ordering — "topological sort":
- Start with set S and partial ordering ρ on S. Idea is to turn S into a sequence $x_1, x_2, \ldots$ such that $(x_i \; \rho \; x_j) \;\rightarrow\; (i \leq j)$.
- The algorithm might look like this in pseudocode:

  while (S not empty)
  pick a minimal element x in S
  make it the next element of the sequence and remove it from S
  end while

  ("Minimal" here means an element such that aren't any that are smaller.)
- Does this work? i.e., does it produce an ordering that extends ρ? True if we can be sure that for x and y with $x \; \rho \; y$ x is picked before y.

Slide 9

Functions

- Formal definition: $f : S \to T$ is a subset of $S \times T$, such that for every $s \in S$, there's *exactly one* (s, t) in the subset. Write $f(s) = t$.
- Terminology: S is f's *domain*. T is f's *co-domain* (or *range*).
- Examples:
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^2$.
  - $g : \mathbb{N} \to \mathbb{R}$ defined by $g(x) = \sqrt{x}$.
  - $h : P \to (P \times P)$ (where P is the set of people in the world) defined by $h(x)$ = ((bio?)mother of x, (bio?)father of x).
- Idea easily extends to functions of more than one variable.

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Properties of Functions

- For $f : S \to T$, f is *onto* if for every $t \in T$ there's an $s \in S$ with $f(s) = t$. "f covers everything in T."
- For $f : S \to T$, f is *one-to-one* if for every $s, s' \in S$, $f(s) = f(s') \;\to\; s = s'$. "$f$ maps different things in S to different things in T".
- If f is both one-to-one and onto, call it a *bijection*.

Composition of Functions

- For $f : S \to T$ and $g : T \to U$, can define $g \circ f :? \to ?$:
  $(g \circ f)(s) = g(f(s))$.

Slide 11

Function Inverses

- If f is a bijection, can define *inverse* of f, $f^{-1} : T \to S$ such that
  $f^{-1} \circ f$ = identity function on S
  $f \circ f^{-1}$ = identity function on T
- Can we do this if f is not a bijection?

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Slide 13

Set Cardinality, Revisited

- We can say that sets S and T have the same cardinality ("same size") if there is a bijection $f : S \to T$ — more formal/precise version of earlier definition, works for both finite and infinite sets.
- If we can define a one-to-one $f : S \to T$, then the cardinality of S is less than or equal to the cardinality of T.
- Recall that we had a "smallest" infinite set $\mathbb{N}$, and a strictly "larger" infinite set $\mathbb{R}$. Are there any bigger sets?

  Yes. Recall that if S is finite with n elements, $\mathscr{P}(S)$ is strictly bigger (2^n elements). True for infinite sets as well — Cantor's theorem.
- Notice that this defines an equivalence relation on sets.

Slide 14

Minute Essay

- None — quiz.