IDENTIFICATION AND CORRECTION OF A COMMON

ERROR IN GENERAL ANNUITY CALCULATIONS

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ABSTRACT

This article examines the conventional method of solving general annuity problems in which general annuities are converted into equivalent simple annuities, thereby enabling standard solution routines. It finds that the method errs when the frequency of payments exceeds the frequency of interest compounding and interest accrues linearly between conversion dates. It demonstrates, and advocates the adoption of, an alternative method, which achieves valid results in such circumstances.

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INTRODUCTION

We define a general annuity as an annuity in which the frequency of payments p differs from the frequency of interest compounding m. For our present purposes we assume that p and m are integers, of which either is a factor of the other, obtained from the same time interval; typically, although not necessarily, one year. Given these constraints, current techniques for solving general annuity problems typically follow a method pioneered by Hummell and Seebeck in the early 1940s. General annuities are converted into equivalent simple annuities, thereby enabling conventional solution routines [Hummell & Seebeck, 1948].

In the preface to the first edition of *Mathematics of Finance* the authors claim that their book "differs widely from existing texts in the treatment of general annuities" and that they have used their advocated method in classroom teaching over the preceding eight years with "gratifying success" [Hummell & Seebeck, 1948, v]. Bruck [1949, 487] comments favourably on the technique, which he describes as "avoiding a whole complex of complicated formulas".

In the preface to the second edition of *Mathematics of Finance*, Hummell and Seebeck [1956, v] claim that their approach to solving general annuity problems has been adopted by "several other writers". They also profess a belief, "shared with many colleagues throughout the country", that theirs is "the only teachable method of treating general annuities".

Hummell and Seebeck convert general annuities into simple annuities by calculating an equivalent payment per interest compounding period. Contemporary finance and financial mathematics textbooks [see, for example, *Mathematics of Finance* by Knox, Zima & Brown, 1999] may alternatively calculate an equivalent interest rate per payment period. Either way, the general annuity is converted into a simple annuity, thereby enabling conventional solution routines.

Despite their provenance and unchallenged acceptance, both methods conventionally used to convert general annuities into simple annuities err when $p > m$ and interest accrues linearly between conversion dates. This article explains that error and how to avoid it.

IDENTIFICATION, EXPLICATION & CORRECTION OF THE ERROR

Given an ordinary general annuity and the constraints previously stated, Equations (1) and (2) ostensibly define an equivalent interest rate i' per payment period and an equivalent payment P' per interest compounding period.

$$i' = (1 + i)^{m/p} - 1 \tag{1}$$

$$P' = P\frac{i}{i'} \tag{2}$$

where i = interest rate per compounding period

P = payment.

We can use either equivalency to convert general annuities into simple annuities and then use simple annuity solution methods to solve general annuity problems. For example, Equations (3A/B) and (4A/B) use respectively i' and P' to define the present and future values of an ordinary general annuity.

$$PV = P\left[\frac{1 - \left(1 + i'\right)^{-n}}{i'}\right] \tag{3A}$$

$$FV = P\left[\frac{\left(1 + i'\right)^{n} - 1}{i'}\right] \tag{3B}$$

$$PV = P'\left[\frac{1 - \left(1 + i\right)^{-n}}{i}\right] \tag{4A}$$

$$FV = P'\left[\frac{\left(1 + i\right)^{n} - 1}{i}\right] \tag{4B}$$

where n = number of interest compounding periods.

But if $p > m$ and interest accrues linearly between conversion dates, Equations (1) and (2) are invalid. It follows that Equations (3A/B) and (4A/B) are also invalid unless i' and P' are correctly redefined. In this context it is more realistic to identify an equivalent payment per interest compounding period that an equivalent interest rate per payment period.[1] We therefore use Equation (5) to provides a valid definition of P'.

$$P' = Pa\left[1 + i\left(\frac{a-1}{2a}\right)\right] \qquad a > 1 \qquad (5)^2$$

where $a = p/m$ = number of payments per interest conversion period.

When $p > m$, Equation (2) gives a lower value of P' than Equation (5). Equation (6A) defines the relative difference Δ.

$$\Delta = \frac{i}{\left[1 + 0.5i\left(a-1\right)\right]\left[\left(1+i\right)^{1/a} - 1\right]} - 1 \qquad a > 1 \qquad (6A)$$

The same relative difference applies to present and future values derived from the alternative definitions of P'.

The following examples demonstrate how conventional approaches to solving general annuity problems err when $p > m$ and interest accrues linearly between conversion dates. We use the same examples to demonstrate the advocated alternative approach, which correctly solves these problems.

Example 1

An ordinary general annuity comprises $3,000 paid quarterly for ten years at an annual interest rate of 8%, compounded annually. What is its future value?

Values of the relevant variables are therefore:

$$P = \$3,000$$

$$i = 8\%$$

$$m = 1$$

$$p = 4$$

$$a = p/m = 4/1 = 4$$

$$n = 10$$

Equation (5) correctly defines the value of P' to be \$12,360, as follows:

$$P' = Pa\left[1 + i\left(\frac{a-1}{2a}\right)\right] \qquad a > 1 \qquad (5)$$

$$= \$3,000 \times 4 \times \left[1 + 0.08\left(\frac{4-1}{8}\right)\right]$$

$$= \$3,000 \times 4.12$$

$$= \$12,360$$

Table I confirms the foregoing figure by identifying and summing the interest earned by each payment between conversion dates.[3]

Table I: Summary of interest earned by four quarterly payments of $3,000 at an effective annual rate of 8% compounded annually

Payment no.	Amount	Months of interest	Interest earned	
			Algebraic notation	$
1	$3,000	9	$Pi(3/a)$	180
2	$3,000	6	$Pi(2/a)$	120
3	$3,000	3	$Pi(1/a)$	60
4	$3,000	0		0
Total interest earned				360

Equations (1) and (2) erroneously define the value of P' as $12,354.23, which is approximately 0.0467% below the correct figure, as follows:

$$i' = (1 + i)^{m/p} - 1 \tag{1}$$

$$= (1.08)^{1/4} - 1$$

$$= 1.942654691\%$$

$$P' = P\frac{i}{i'} \tag{2}$$

$$= \$3,000 \times \frac{0.08}{0.01942654691}$$

$$= \$12,354.23$$

Table II confirms the foregoing figure by identifying and summing the interest incorrectly assumed to be earned by each payment between conversion dates.

Table II: Summary of interest earned by four quarterly payments

of $3,000 at an effective annual rate of 8% compounded quarterly

Payment no.	Amount	Months of interest	Interest earned	
			Algebraic notation	$
1	$3,000	9	$P[(1 + i)^3 - 1]$	178.26
2	$3,000	6	$P[(1 + i)^2 - 1]$	117.69
3	$3,000	3	Pi	58.28
4	$3,000	0		0
Total interest earned				354.23

Equation (6A) confirms a relative difference of −0.0467%, as follows:

$$\Delta = \frac{i}{\left[1 + 0.5i\left(a - 1\right)\right]\left[\left(1 + i\right)^{1/a} - 1\right]} - 1 \qquad a > 1 \qquad (6A)$$

$$= \frac{i}{\left[1 + 0.04\left(4 - 1\right)\right]\left(.08^{1/4} - 1\right)} - 1$$

$$= \frac{0.08}{\left(1.12\right)\left(0.0194265469\right)} - 1$$

$$= -0.000467$$

An error of less than 0.05% may seem insignificant, but present and future values derived from the competing methods exhibit the same relative error, the significance of which is thereby amplified. Using the present example, we can confirm that either of the conventional solution methods understates the future value by $83.61.

$$FV = P\left[\frac{\left(1 + i\right)^n - 1}{i'}\right] \qquad (3B)$$

$$= \$3{,}000 \left[\frac{1.08^{10} - 1}{0.01942654691} \right]$$

$$= \$3{,}000 \times 59.65676776$$

$$= \$178{,}970.30$$

$$FV = P' \left[\frac{\left(1 + i\right)^n - 1}{i} \right] \tag{4B}$$

$$= \$12{,}354.229 \left[\frac{1.08^{10} - 1}{0.08} \right]$$

$$= \$12{,}354.229 \times 14.48656247$$

$$= \$178{,}970.30$$

The correct equivalent annual payment of $12,360 correctly defines a future value of $179,053.91, as follows:

$$\$12{,}360 \times 14.48656247$$

$$= \$179{,}053.91$$

Financial Calculator Solution

Financial calculators typically compound interest continuously when solving general annuity problems. This produces valid results when $p < m$, but not when $p > m$ and interest accrues linearly between conversion dates. For example, in the current case the Sharp EL-738 calculator computes an incorrect future value of $178,970.30.

Example 2

A loan of $1million is to be repaid in equal monthly instalments over four years. If the annual interest rate is 10% compounded semi-annually, how much is the monthly repayment?

Conventional Solution

$$i' = (1+i)^{m/p} - 1 \tag{1}$$

$$= 1.05^{2/12} - 1$$

$$= 0.8164846\%$$

$$PV = P\left[\frac{1-(1+i)^{-n}}{i'}\right] \tag{3A}$$

$$\therefore \ P = PV\left[\frac{i'}{1-(1+i)^{-n}}\right]$$

$$= \$1,000,000\left[\frac{0.008164846}{1-1.05^{-8}}\right]$$

$$= \$1,000,000 \times 0.025265595$$

$$= \$25,265.60$$

But if interest accrues linearly between conversion dates, monthly repayments of $25,265.60 over four years at an annual interest rate of 10% compounded semi-annually will repay $193.64 more than $1million, as shown by Equation (7A) and confirmed in the Appendix to this article.

$$PV = P'\left[\frac{1-(1+i)^{-n}}{i}\right] \tag{4A}$$

$$= Pa \left[1 + i \left(\frac{a-1}{2a} \right) \right] \left[\frac{1 - (1+i)^{-n}}{i} \right] \qquad a > 1 \qquad \text{(7A)}$$

$$= \$25,265.60 \times 6 \times \left[1 + 0.5 \left(\frac{6-1}{12} \right) \right] \left[\frac{1 - 1.05^{-8}}{0.05} \right]$$

$$= \$25,265.60 \times 6.125 \times 6.463212759$$

$$= \$1,000,193.64$$

In this example the conventional solution method has generated a positive error of 0.019364%, as confirmed by Equation (6B).

$$\Delta = \frac{\left[1 + 0.5i (a-1) \right] \left[(1+i)^{1/a} - 1 \right]}{i} - 1 \qquad a > 1 \qquad \text{(6B)}$$

$$= \frac{\left[1 + 0.025(6-1) \right] \left(.05^{1/6} - 1 \right)}{0.05} - 1$$

$$= \frac{0.050009682}{0.05} - 1$$

$$= 0.00019364$$

Advocated Solution

Equation (7B) correctly defines a monthly repayment of \$25,260.70, as follows:

$$\text{PV} = Pa \left[1 + i \left(\frac{a-1}{2a} \right) \right] \left[\frac{1 - (1+i)^{-n}}{i} \right] \qquad a > 1 \qquad \text{(7A)}$$

$$\therefore P = \text{PV} \left[\frac{i}{1 - (1+i)^{-n}} \right] \div a \left[1 + i \left(\frac{a-1}{2a} \right) \right] \qquad a > 1 \qquad \text{(7B)}$$

$$= \$1,000,000 \left[\frac{0.05}{1 - 1.05^{-8}} \right] \div 6 \left[1 + 0.05 \left(\frac{5}{12} \right) \right]$$

$$= \$1,000,000 \times 0.154721813 \div 6.125$$

$$= \$25,260.70$$

Equation (7A) and the Appendix confirm that monthly payments of $25,260.70 over four years at an annual interest rate of 10% compounded semi-annually will repay exactly $1million; as follows:

$$\text{PV} = Pa\left[1+i\left(\frac{a-1}{2a}\right)\right]\left[\frac{1-\left(1+i\right)^{-n}}{i}\right] \qquad a > 1 \qquad (7A)$$

$$= \$25,260.70 \times 6 \times \left[1+0.05\left(\frac{6-1}{12}\right)\right]\left[\frac{1-1.05^{-8}}{0.05}\right]$$

$$= \$25,260.70 \times 6.125 \times 6.463212759$$

$$= \$1,000,000$$

Calculator Solution

Applied to this example, the Sharp EL-738 calculator computes a monthly payment of $25,265.60, which is incorrect when interest accrues linearly between conversion dates.

ERROR ANALYSIS

Examples 1 and 2 confirm that, when $p > m$ and interest accrues linearly between conversion dates, the conventional approach to solving general annuity problems understates the present value of a given payment stream and overstates the periodic amount required to repay a given loan

Figure 1 illustrates the relative magnitude of that understatement of present and future values over a range of interest rates and payments per interest conversion period.
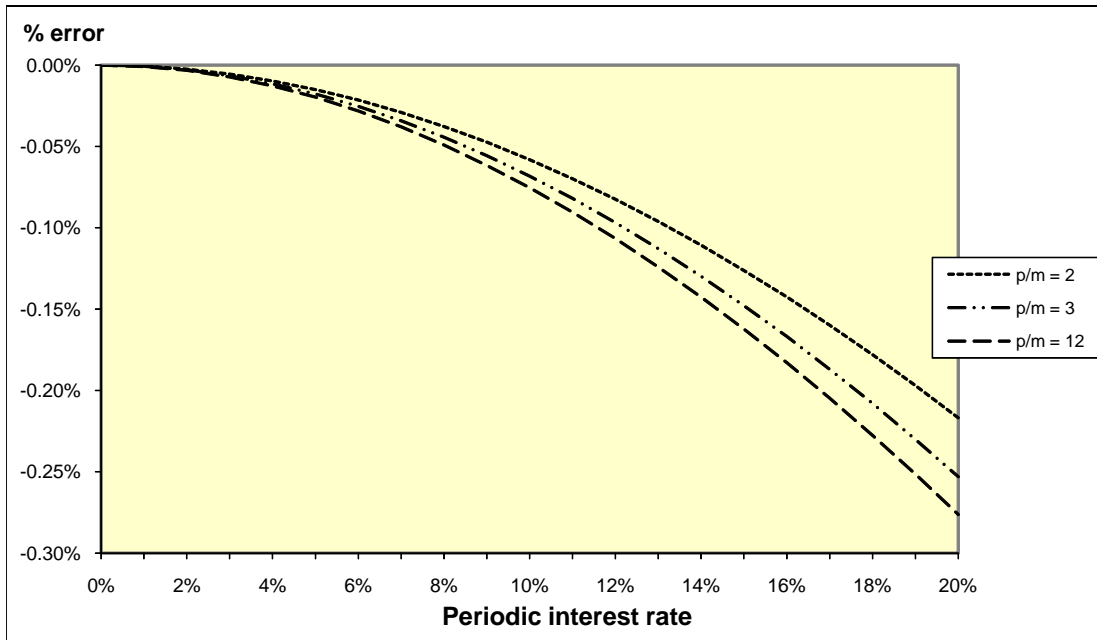

Figure 1. Percentage errors produced by conventional calculations of the present and future values of general annuities when $p > m$ and interest accrues linearly between conversion dates

Figure 1 confirms that the periodic interest rate is the dominant factor in distorting present and future values when conventional solution techniques are applied to general annuity problems in which $p > m$ and interest accrues linearly between conversion dates. Errors are invariant to the term of an annuity. They can be insignificant in individual cases, but acquire significance through aggregation. For example, a financial intermediary could gain a significant advantage when loan repayments are structured as general annuities in which $p > m$ and periodic repayments are calculated in the conventional manner.

ADVOCACY FOR CHANGE

Do these findings justify changing the way we currently conceptualise and teach the solving of general annuity problems when $p > m$? Before answering that question, we need to resolve a concomitant issue: how does interest actually accrue between conversion dates? My own research confirms that interest typically accrues linearly between conversion dates, as depicted in Table I, rather than exponentially, as depicted in Table II and subsumed in financial calculators. That being the case, the conventional approach to solving general annuity problems when $p > m$ is founded upon a mistaken assumption. I therefore advocate a shift towards reality in our teaching of general annuities. Mistaken assumptions that become known pretences are indefensible bases for pedagogical methodologies, however well entrenched.

SUMMARY & CONCLUSION

This article demonstrates how the conventional approach to solving general annuity problems is flawed when the frequency of payments exceeds the frequency of interest compounding and interest accrues linearly between conversion dates. It identifies the source of error and proposes an alternative solution technique, which produces valid results. It finds that the relative errors generated by conventional solution methods are low (typically less than 0.2%), but contends that the use of flawed routines cannot be justified when more robust alternatives are available.

<div align="center">*   *   *   *</div>

ENDNOTES

[1] When $p > m$ and interest accrues linearly between conversion dates, we need to define i' as follows in order to validate Equations (3A/B):

$$i' = \frac{i}{a\left[1 + i\left(\dfrac{a-1}{2a}\right)\right]} \qquad\qquad a > 1 \qquad\qquad (8)$$

where $a = p/m$

However, we should not recommend this approach, because, contrary to the implication of Equations (3A/B), interest is not compounded between conversion dates. Furthermore, the rate of interest per payment period is actually i/a. Equation (8) therefore defines a spurious value of i', which validates Equations (3A/B), but is unobservable.

[2] We can adapt Equation (5) to define P' for general annuities in which cash flows occur either at the start or in the middle of each payment period, as follows:

$$P' = Pa\left[1 + i\left(\frac{a+1}{2a}\right)\right] \qquad\qquad a > 1 \qquad\qquad (5A)$$

$$P' = Pa(1 + 0.5i) \qquad\qquad a > 1 \qquad\qquad (5B)$$

[3] Applied to the same example, Equations (5A) and (5B) define equivalent payments per interest conversion period of \$12,600 and \$12,480 respectively. If payments occur at the beginning of each quarter, the first, second, third and fourth payments attract interest of \$240, \$180, \$120 and \$60 respectively. Similarly, payments made in the middle of each quarter earn interest of \$210, \$150, \$90 and \$30 per successive payment. A continuous payment stream achieves the same outcome.

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APPENDIX: EXAMPLES OF LOAN REPAYMENT SCHEDULES

	I N P U T						O U T P U T	
APR	10.00%	*m*	2	*p*	12	EAR	10.25%	

	Assuming continuous compounding		Assuming linear accrual of interest between conversion dates					
	PMT	Cl. bal.	PMT	Interest	Cl. Bal.	PMT	Interest	Cl. Bal.
0		-1,000,000.00			-1,000,193.64			-1,000,000.00
1	25,265.60	-982,899.25	25,265.60	-8,334.95	-974,928.04	25,260.70	-8,333.33	-974,739.30
2	25,265.60	-965,658.88	25,265.60	-8,124.40	-949,662.45	25,260.70	-8,122.83	-949,478.59
3	25,265.60	-948,277.74	25,265.60	-7,913.85	-924,396.85	25,260.70	-7,912.32	-924,217.89
4	25,265.60	-930,754.68	25,265.60	-7,703.31	-899,131.26	25,260.70	-7,701.82	-898,957.18
5	25,265.60	-913,088.55	25,265.60	-7,492.76	-873,865.66	25,260.70	-7,491.31	-873,696.48
6	25,265.60	-895,278.19	25,265.60	-7,282.21	-895,451.55	25,260.70	-7,280.80	-895,278.19
7	25,265.60	-877,322.40	25,265.60	-7,462.10	-870,185.95	25,260.70	-7,460.65	-870,017.48
8	25,265.60	-859,220.01	25,265.60	-7,251.55	-844,920.36	25,260.70	-7,250.15	-844,756.78
9	25,265.60	-840,969.81	25,265.60	-7,041.00	-819,654.76	25,260.70	-7,039.64	-819,496.07
10	25,265.60	-822,570.60	25,265.60	-6,830.46	-794,389.16	25,260.70	-6,829.13	-794,235.37
11	25,265.60	-804,021.17	25,265.60	-6,619.91	-769,123.57	25,260.70	-6,618.63	-768,974.66
12	25,265.60	-785,320.28	25,265.60	-6,409.36	-785,472.35	25,260.70	-6,408.12	-785,320.28
13	25,265.60	-766,466.71	25,265.60	-6,545.60	-760,206.76	25,260.70	-6,544.34	-760,059.58
14	25,265.60	-747,459.19	25,265.60	-6,335.06	-734,941.16	25,260.70	-6,333.83	-734,798.87
15	25,265.60	-728,296.49	25,265.60	-6,124.51	-709,675.56	25,260.70	-6,123.32	-709,538.17
16	25,265.60	-708,977.32	25,265.60	-5,913.96	-684,409.97	25,260.70	-5,912.82	-684,277.46
17	25,265.60	-689,500.41	25,265.60	-5,703.42	-659,144.37	25,260.70	-5,702.31	-659,016.76
18	25,265.60	-669,864.48	25,265.60	-5,492.87	-669,994.19	25,260.70	-5,491.81	-669,864.48
19	25,265.60	-650,068.23	25,265.60	-5,583.28	-644,728.60	25,260.70	-5,582.20	-644,603.78
20	25,265.60	-630,110.34	25,265.60	-5,372.74	-619,463.00	25,260.70	-5,371.70	-619,343.07
21	25,265.60	-609,989.50	25,265.60	-5,162.19	-594,197.41	25,260.70	-5,161.19	-594,082.37
22	25,265.60	-589,704.37	25,265.60	-4,951.65	-568,931.81	25,260.70	-4,950.69	-568,821.66
23	25,265.60	-569,253.62	25,265.60	-4,741.10	-543,666.22	25,260.70	-4,740.18	-543,560.96
24	25,265.60	-548,635.89	25,265.60	-4,530.55	-548,742.13	25,260.70	-4,529.67	-548,635.89
25	25,265.60	-527,849.82	25,265.60	-4,572.85	-523,476.53	25,260.70	-4,571.97	-523,375.19
26	25,265.60	-506,894.04	25,265.60	-4,362.30	-498,210.94	25,260.70	-4,361.46	-498,114.48
27	25,265.60	-485,767.16	25,265.60	-4,151.76	-472,945.34	25,260.70	-4,150.95	-472,853.78
28	25,265.60	-464,467.78	25,265.60	-3,941.21	-447,679.75	25,260.70	-3,940.45	-447,593.07
29	25,265.60	-442,994.49	25,265.60	-3,730.66	-422,414.15	25,260.70	-3,729.94	-422,332.37
30	25,265.60	-421,345.87	25,265.60	-3,520.12	-421,427.46	25,260.70	-3,519.44	-421,345.87
31	25,265.60	-399,520.50	25,265.60	-3,511.90	-396,161.87	25,260.70	-3,511.22	-396,085.17
32	25,265.60	-377,516.93	25,265.60	-3,301.35	-370,896.27	25,260.70	-3,300.71	-370,824.46
33	25,265.60	-355,333.70	25,265.60	-3,090.80	-345,630.67	25,260.70	-3,090.20	-345,563.76
34	25,265.60	-332,969.35	25,265.60	-2,880.26	-320,365.08	25,260.70	-2,879.70	-320,303.06
35	25,265.60	-310,422.40	25,265.60	-2,669.71	-295,099.48	25,260.70	-2,669.19	-295,042.35
36	25,265.60	-287,691.35	25,265.60	-2,459.16	-287,747.06	25,260.70	-2,458.69	-287,691.35
37	25,265.60	-264,774.71	25,265.60	-2,397.89	-262,481.47	25,260.70	-2,397.43	-262,430.65
38	25,265.60	-241,670.96	25,265.60	-2,187.35	-237,215.87	25,260.70	-2,186.92	-237,169.94
39	25,265.60	-218,378.57	25,265.60	-1,976.80	-211,950.27	25,260.70	-1,976.42	-211,909.24
40	25,265.60	-194,896.00	25,265.60	-1,766.25	-186,684.68	25,260.70	-1,765.91	-186,648.54
41	25,265.60	-171,221.70	25,265.60	-1,555.71	-161,419.08	25,260.70	-1,555.40	-161,387.83
42	25,265.60	-147,354.11	25,265.60	-1,345.16	-147,382.64	25,260.70	-1,344.90	-147,354.11
43	25,265.60	-123,291.64	25,265.60	-1,228.19	-122,117.04	25,260.70	-1,227.95	-122,093.40
44	25,265.60	-99,032.70	25,265.60	-1,017.64	-96,851.45	25,260.70	-1,017.45	-96,832.70
45	25,265.60	-74,575.69	25,265.60	-807.10	-71,585.85	25,260.70	-806.94	-71,571.99
46	25,265.60	-49,918.99	25,265.60	-596.55	-46,320.26	25,260.70	-596.43	-46,311.29
47	25,265.60	-25,060.98	25,265.60	-386.00	-21,054.66	25,260.70	-385.93	-21,050.59
48	25,265.60	**0.00**	25,265.60	-175.46	**0.00**	25,260.70	-175.42	**0.00**