

## RECONCILIATION OF LEARNING RATES BETWEEN THE CUMULATIVE AVERAGE VS INCREMENTAL LEARNING RATE MODELS

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**Abstract** — There are two basic learning curve time and cost models that are somewhat confusing, because the same learning rate (e.g. the 80% learning curve) yields different results. The cumulative average model applies the learning rate to average variable cost, whereas the incremental model applies it to marginal cost. This note stresses that, even though an analyst may prefer to conceptualize learning as an explicit constant in the incremental unit learning model, that same analyst has implicitly assumed a varying learning rate in the cumulative average model. A vice versa situation arises if a constant learning rate is assumed in the cumulative average model. Textbooks that compare both models do not reconcile them in the manner presented in this paper.

**Key words:** Management accounting, cost accounting, learning curves, cost models.

### 1. TWO POPULAR LEARNING CURVE MODELS IN TEXTBOOKS

In 1936, T. P. Wright described how average unit cost in aircraft manufacturing declined due to learning effects of repeated production applications. Furthermore, the learning effects seemed to be somewhat of a stationary process on what became widely known as the 80% non-linear learning curve. A historical review of learning curve analysis is presented by Yelle (1979). A survey of various methods is also given in Belkaoui (1986). Arrow (1962) discusses implications in economic theory. Learning curve analysis is covered in virtually all textbooks in cost and managerial accounting, e.g. Anthony and Reece (1989) and Horngren and Foster (1991). The topic appears in texts of other disciplines such as operations research by Hillier and Lieberman (1987) and managerial economics by Peppers and Bails (1987).

Recently, Pattison and Teplitz (1989) find learning curve analysis useful in flexible manufacturing systems (FMS), thereby countering doubts raised about its usefulness in FMS by Bylinsky (1983). Learning curve analysis is purportedly used in contract pricing decisions, e.g. see Pegels (1976) and Morse (1972). Use in industrial accident analysis is reported by Greenburg (1970) and warranty risks by Kneip (1965). Applications in financial planning are proposed by Harvey (1976) following up on earlier uses suggested in Summers and Welsch (1970). Use in planning and forecasting includes the works of Conley (1970), Nathanson (1972), and Yelle (1974, 1976). Abernathy and Wayne (1974) use learning curve analysis in a historical comparison of Model A vs Model T Fords in a business policy context.

Some textbooks in cost/managerial accounting present two learning curve models that

differ in terms of whether the learning rate applies to the average variable cost vs the marginal unit cost, for example Kaplan (1982, pp. 96–107) contrasts the two models (that are variations on Wright's original 1936 model). In spite of many proposed alternatives, these particular models remain the focal points of most textbooks and real-world applications. For example, Problems 10–19 and 10–20 in Horngren and Foster (1987, pp. 369–370) show how a  $C = 80\%$  cumulative average model learning rate yields a dramatically different make-vs-buy decision for oil rigs of the Oceanic Exploration Company than does an  $I = 80\%$  incremental unit model learning rate as the quantity,  $q$ , is increased. A somewhat similar exposition of both models is given in Hirsch (1988, pp. 156–175).

The purpose of this note is to show how the implicit  $C = C(I, q)$  cumulative average learning rate can be derived from an explicit  $I$  incremental learning parameter or vice versa for deriving an implicit  $I = I(C, q)$  incremental model learning rate from an explicit  $C$  cumulative average model learning parameter. This serves to emphasize that two different learning rates,  $I(C, q)$  and  $C(I, q)$ , are really being assumed no matter which learning curve model (cumulative average vs incremental unit) is specified. This note provides a quick and easy way to relate one rate to the other in a given model.

The notation and key learning rate model equations are as follows if  $q$  units of product or service are sequentially produced:

#### I. Cumulative average learning rate model

$$C \equiv \text{Learning rate (in \% ) on average variable cost.} \quad (1)$$

$$c \equiv \text{Learning rate exponent} = [\ln (c/100)]/(\ln 2). \quad (2)$$

$$u(1) \equiv \text{Marginal cost of the first unit.} \quad (3)$$

$$\bar{v}(q) \equiv \text{Average variable cost of } q \text{ units} = u(1) q^c. \quad (4)$$

$$V(q) \equiv \text{Cumulative variable cost of } q \text{ units} = q\bar{v}(q). \quad (5)$$

$$\begin{aligned} u(j) \equiv \text{Marginal cost of unit } j &= V(j) - V(j-1) \\ &= u(1) [j^{1+c} - (j-1)^{1+c} + (j-1)^c]. \end{aligned} \quad (6)$$

#### II. Incremental unit learning rate model

$$I \equiv \text{Learning rate (in \% ) on incremental marginal cost.} \quad (7)$$

$$i \equiv \text{Learning rate exponent} = (\ln (I/100))/(\ln 2). \quad (8)$$

$$u(j) \equiv \text{Marginal unit cost} = u(1)j^i, \text{ for } j = 1, \dots, q. \quad (9)$$

$$V(q) \equiv \text{Cumulative variable cost} = \sum_{j=1}^q u(j). \quad (10)$$

$$\bar{v}(q) \equiv \text{Average variable cost} = (1/q) V(q). \quad (11)$$

### 2. RELATING $C(I, q)$ TO $I$

Suppose  $V(q)$  is the cumulative variable cost of  $q$  units under the incremental unit model with a learning rate  $I$ . The  $C(I, q)$  learning rate corresponding to  $I$  and  $q$  is such that for  $c = c(i, q)$

$$V(q) = (q)u(1)q^{c(i,q)} = \sum_{j=1}^q u(j). \quad (12)$$

After some algebraic manipulation, the  $c(i,q)$  exponent can be derived in closed form as follows:

$$c(i,q) = -1 + \left[ \frac{\ln \left( \sum_{j=1}^q j^i \right)}{\ln (q)} \right]. \quad (13)$$

For any exponent  $y$ , we will define  $\exp (y)$  as

$$e^y = \exp (Y) \approx (2.17828183)^y. \quad (14)$$

Recalling that  $c(i,q) = \ln (C/100)/\ln 2$ , we may then perform some additional algebra to obtain  $C(I,q)$  in closed form as

$$C(I,q) = (100) \exp \left\{ (\ln 2) \left[ (-1) + \frac{\ln \left( \sum_{j=1}^q j^i \right)}{\ln (q)} \right] \right\}. \quad (15)$$

Values of  $C(I,q)$  cumulative average learning rates (for selected values of  $q$  and  $I = 80\%$  constant incremental unit model learning rate) are shown in Table 1. The  $C(I,q)$  cumulative average rate always exceeds the  $I$  constant learning rate, but as  $q$  output approaches infinity,  $C(I,q)$  approaches  $I$  asymptotically. But it is important to note how slowly the cumulative average rate approaches the incremental unit rate. In Table 1, for example, when  $q = 20$ , the cumulative average rate of 86.12% is not even half way in its descent from 90% (for  $q = 2$ ) down to 80% (when  $q$  is infinite). A graph of the approach of the cumulative average rate toward the constant incremental rate is shown in Fig. 1.

Also in Table 1, the vice versa relationship is shown when the  $C = 80\%$  cumulative average learning rate is held constant. In this case, the  $I(C,q)$  incremental learning rate is variable when  $C$  is held constant. As output  $q$  increases, the incremental learning rate moves upward asymptotically toward the cumulative average rate. In Table 1 when  $C = 80\%$  constant, the  $I(80\%, q)$  rate begins at 60% when  $q = 1$  and moves toward 80% relatively slowly.

### 3. AN ILLUSTRATION UNDER THE $I = 80\%$ LEARNING CURVE

Suppose an incremental unit learning curve model has the following specifications and outcomes:

$$I = 80\% \text{ Learning rate on incremental marginal cost.}$$

Table 1. Comparisons of cumulative average learning rates  $C(I, q)$  as functions of incremental unit learning rates  $I$  and  $I(C, Q)$  as functions of  $C$ 

Number of units $q =$	When $I = (C, q) = 80\%$ constant, $C(80, q)$ varies		When $C = (I, q) = 80\%$ constant, $I(80, q)$ varies	
	$I(C, q) =$	$C(80, q) =$	$I(C, q) =$	$C(80, q) =$
2	80%	90.00%	60.00%	80%
3	80	89.18	62.79	80
4	80	88.63	64.50	80
5	80	88.22	65.68	80
6	80	87.90	66.55	80
7	80	87.64	67.23	80
8	80	87.43	67.78	80
9	80	87.24	68.25	80
20	80	86.12	70.82	80
30	80	85.64	71.82	80
40	80	85.33	72.43	80
50	80	85.11	72.86	80
60	80	84.93	73.18	80
70	80	84.79	73.44	80
80	80	84.68	73.64	80
90	80	84.58	73.82	80
100	80	84.50	73.97	80
200	80	83.99	74.82	80
300	80	83.74	75.23	80
400	80	83.58	75.50	80
500	80	83.46	75.68	80
Infinite	80	80.00	80.00	80

$$i \approx -0.321928 \text{ learning exponent} = (\ln (0.80/100))/(\ln 2).$$

$$u(1) = \$1,000,000 \text{ marginal cost of the first production unit.}$$

$$u(8) = u(1) (8)^i \approx \$512,000 \text{ marginal cost of the } q = 8\text{th unit.}$$

$$V(8) = \sum_{j=1}^8 u(j) \approx \$5,345,914 \text{ cost of 8 units.}$$

$$\bar{v}(8) = (1/8)/V(8) \approx \$664,489 \text{ average cost per unit.}$$

Students and/or managers may be interested in what cumulative average model learning rate would achieve the same  $V(8) \approx \$5,345,914$  total cost. Using results in Table 1, we see that for  $I = 80\%$  and  $q = 8$  units, the corresponding cumulative average model learning rate is  $C(80, 8) \approx 87.43\%$ .

Hence the same results can be obtained from the following cumulative average learning rate model:

$$C = 87.43\% \text{ Learning rate on average variable cost.}$$

$$c \approx -0.1937997 \text{ Learning exponent} = [\ln (87.43/100)]/(\ln 2).$$

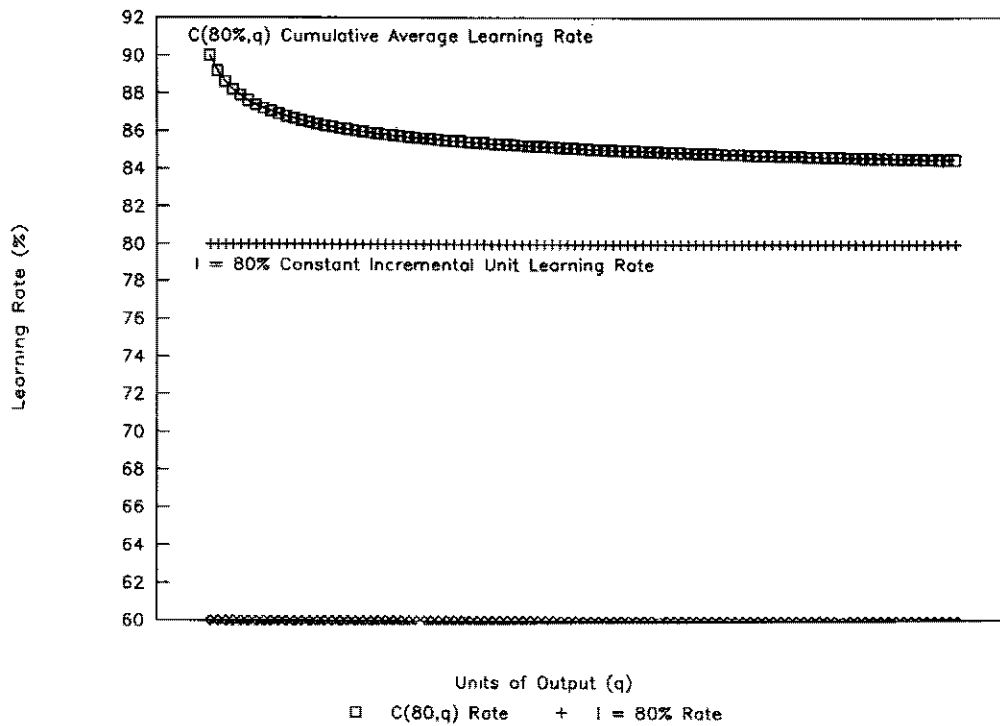


Fig. 1. Cumulative average  $C(I,q)$  learning rate.

$u(1) = \$1,000,000$  marginal cost of the first unit.

$\bar{v}(8) = u(1) (8)^c \approx \$664,489$  average cost per unit.

$V(8) = \sum_{j=1}^8 u(j) \approx \$5,345,914$  cost of 8 units.

$u(8) = V(8) - V(7) \approx \$5,345,914 - \$4,833,914 = \$512,000$ .

#### 4. CONCLUSION

The above results show that, if the equivalent  $I$  and  $C(I,q)$  values are chosen, it makes no difference whether the analyst chooses the incremental learning rate model or the cumulative average learning rate model. The equations in this note show how the equivalent rates can be derived. Choice of one model over another, however, does influence whether  $I$  or  $C$  is constant. In the incremental unit model,  $I$  is constant and  $C = C(I,q)$  varies with the specified  $I$  and  $q$  levels. In the cumulative average model,  $C$  is constant and  $I = I(C,q)$  varies with the specified  $C$  and  $q$  levels.

The  $C(I,q)$  formula is derived in Equation (15) in closed form. The  $I(C,q)$  formula cannot be derived in closed form. If  $C(I,q)$  and  $c(i,q)$  are specified constants, however, values of  $i = i(c,q)$  can be derived iteratively by trial and error in Equation (13). Values of  $I(C,q)$  can then be derived from  $i(c,q)$  amounts using the following formula:

$$I = (100)e^{\{(i)(\ln 2)\}} \quad (16)$$

This note also stresses that, even though an analyst may prefer to conceptualize learning as an explicit constant  $I$  in the incremental unit learning model or a constant  $C$  in the cumulative average learning model, *that same analyst has implicitly assumed a  $C(I, q)$  varying learning rate on the constant  $I$  or a  $I(C, q)$  varying learning rate on the constant  $C$ .* Perhaps learning curve reports would be more meaningful if they disclosed both  $I(C, q)$  and  $C(I, q)$  learning rates assumed in every learning curve analysis.

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