MONOTONIC MAXIMIN:

A ROBUST STACKELBERG SOLUTION AGAINST BOUNDEDLY RATIONAL FOLLOWERS

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Deployed Physical Security Applications

- Limited security resources: selective checking
- Adversary monitors defense, exploits pattern

Stackelberg Games

- Leader (defender) commits to mixed strategy
- Follower (adversary) conducts surveillance and responds

Adversary

	Target #1	Target #2
Target #1	5, -3	-1, 1
Target #2	-5, 5	2, -1

Bounded Rationality

- Strong Stackelberg equilibrium: Classical game theory
  - *Assumes perfect rationality (maximize expected utility)*

- In reality, adversaries are humans

- Quantal Response (McFadden; Mckelvey & Palfrey; Yang et al)

$$q_i(x) = \frac{e^{\lambda U_i^a(x)}}{\sum_j e^{\lambda U_j^a(x)}}$$

  - Need data to estimate parameter

Robust Optimization Approaches

- Uncertainty set: set of possible response functions by the adversary
- Optimize worst-case defender utility
- Allow arbitrary adversary response: Maximin
  - *Robust but very conservative*
- Are there more interesting ways to define uncertainty set that captures bounded-rational behavior?

Monotonic Maximin

- Monotonicity: actions with higher expected utility are played with higher probability
  - QR satisfies monotonicity

- Monotonic maximin: optimize defender utility against worst-case monotonic adversary
  - A robust alternative to QR
  - Provides guarantee against all “reasonably rational” adversary

- Computing monotonic maximin
  - MILP formulation
  - Approximations

Game

- Defender mixed strategy $x \in X \subset \mathbb{R}^m$
  - *X* convex
- Adversary mixed strategy $y \in Y$

$$Y = \{y \in R^n | y \geq 0, 1^T y = 1\}$$

- Payoff Matrices $A, B \in \mathbb{R}^{m \times n}$

- Expected utility $x^T A y$ $x^T B y$

Behavior Models of Adversary

- Logit Quantal Resposne

$$q_i(x) = \frac{e^{\lambda U_i^a(x)}}{\sum_j e^{\lambda U_j^a(x)}}$$

- Regular Quantal Response (Goeree et al)

1. Interiority: $P_j(\boldsymbol{u}) > 0$ for all j.
2. Continuity: $P_j(\boldsymbol{u})$ is continuously differentiable.
3. Responsiveness: $\frac{\partial P_j(\boldsymbol{u})}{\partial u_j} > 0$ for all j.
4. Monotonicity: $u_j > u_k \Rightarrow P_j(\boldsymbol{u}) > P_k(\boldsymbol{u})$ for all j, k.

Monotonic Maximin

Definition 1. *Given $x \in X, y \in Y$, we say y satisfies* closed monotonicity *if for all $i, j \in [n]$, $x^T B e_i \geq x^T B e_j \Rightarrow y_i \geq y_j$.*

- $Q(x) \subseteq Y$ the set of closed monotonic adversary strategies
- Monotonic Maximin:

$$\arg\max_{x \in X} \min_{y \in Q(x)} x^T A y$$

Properties of Monotonic Maximin

- Monotonic maximin exists in all Stackelberg games
- For zero-sum games, coincides with maximin
- Captures all Regular Quantal Response models
  - Worst-case monotonic response is arbitrarily close to worst-case Regular QR
- Captures other model uncertainties, e.g. payoff
  - add i.i.d. noise (smooth, zero mean) to adversary payoff, assuming adversary best responds, the resulting behavior is monotonic

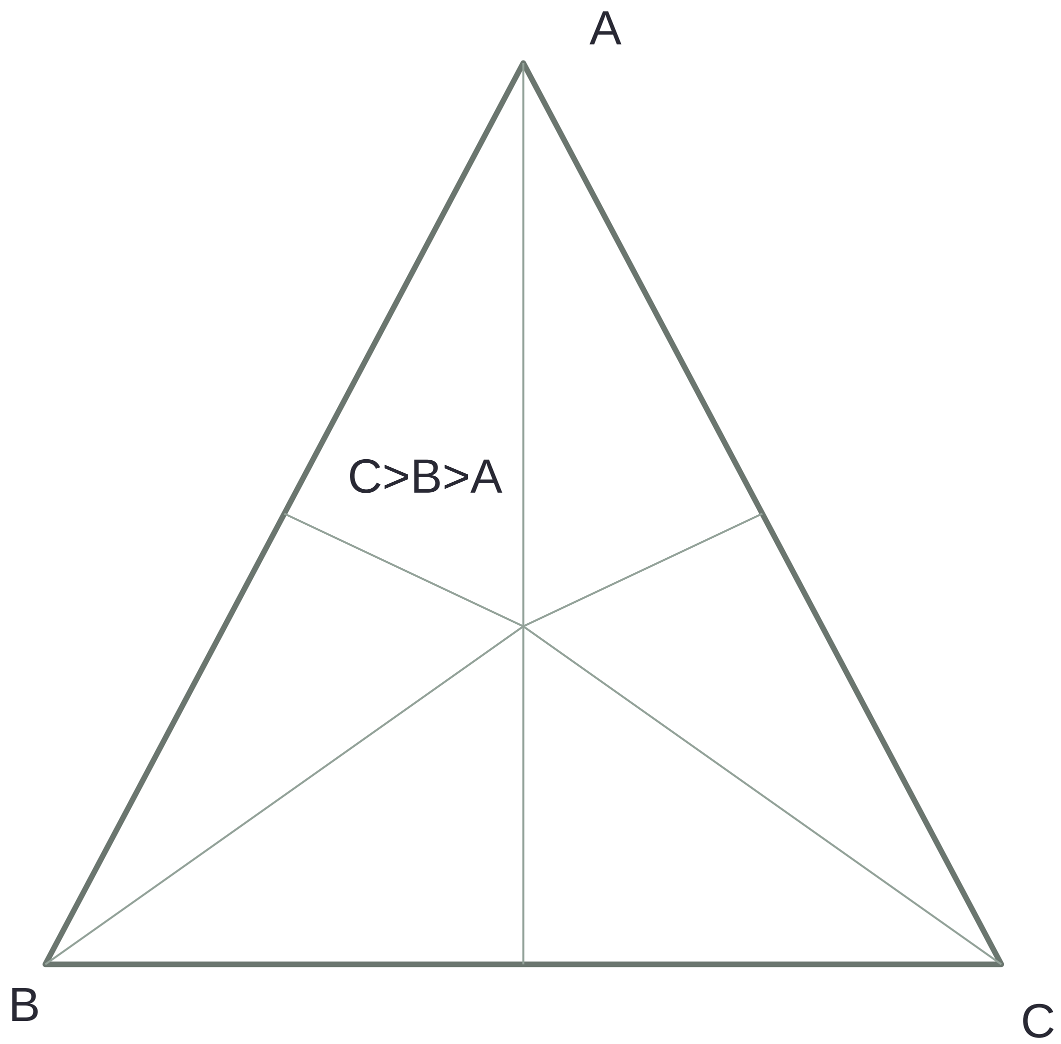
Computation

$$\arg\max_{\boldsymbol{x}\in X} \min_{\boldsymbol{y}\in Q(\boldsymbol{x})} \boldsymbol{x}^T A \boldsymbol{y}$$

- Nontrivial because feasible space of follower depends on leader strategy

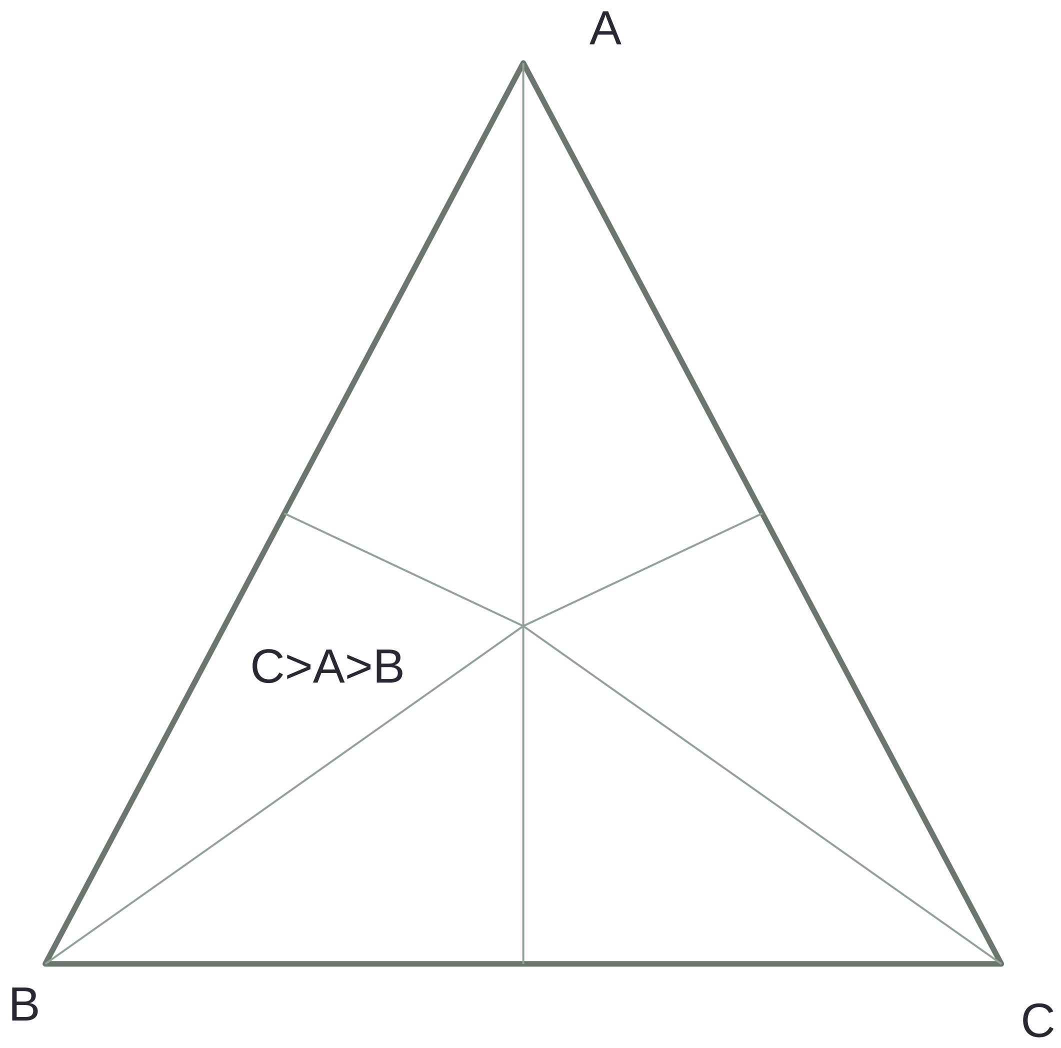
- The set Q(x) depends only on the ordering of actions in terms of adversary expected utilities
  - *Finite # of orderings, thus finite # of possible Q(x)*

Partitioning of leader strategy space X



Corresponding Q(x):
$y_C >= y_B >= y_A$

Partitioning of leader strategy space X



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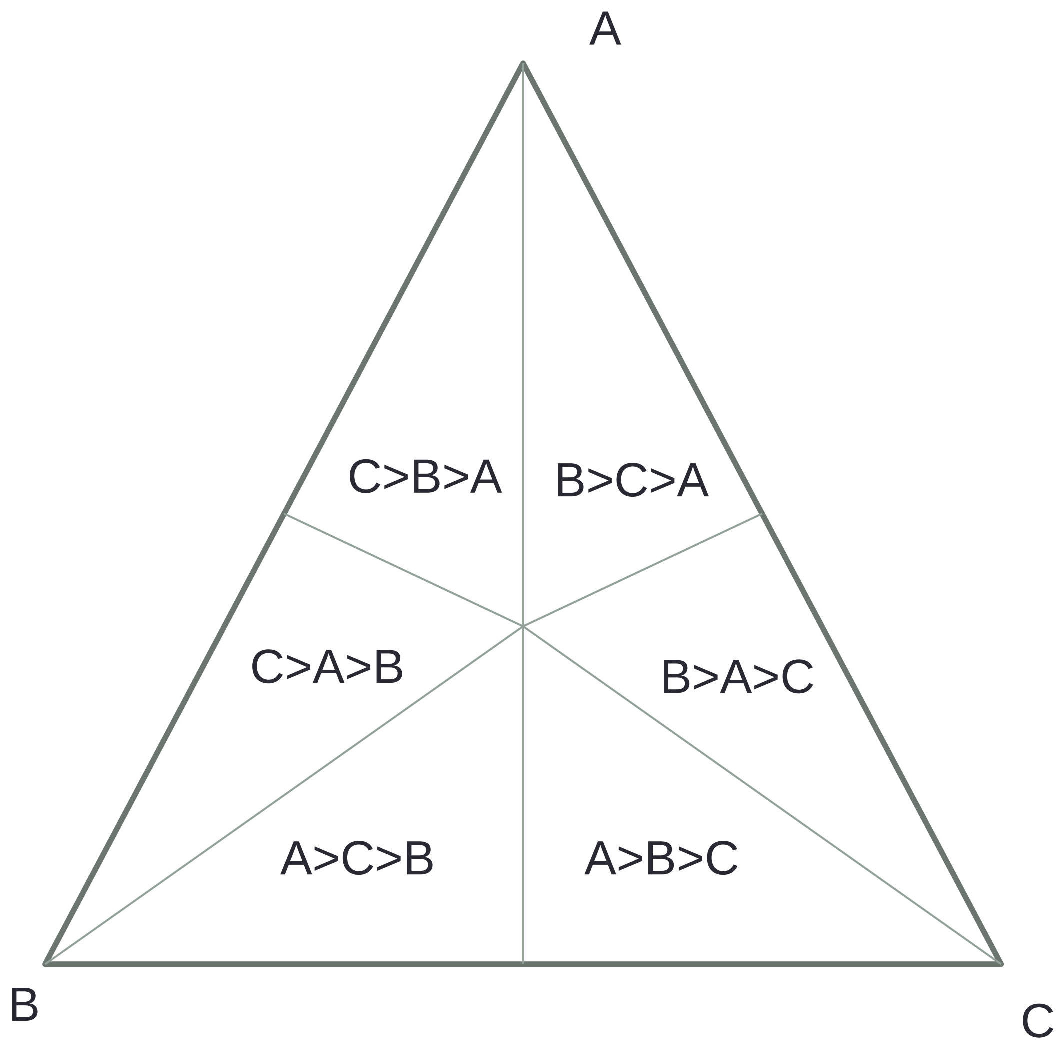
Partitioning of leader strategy space X



Multiple-LP approach

- For each total order on the set of actions, solve

$$\max_{\boldsymbol{x} \in E^{-1}(\mathcal{E})} \min_{\boldsymbol{y} \in Q(x)} x^T A y$$

- Can be formulated as LP

$$\begin{aligned} V_F = \max_{\boldsymbol{x}, \boldsymbol{\lambda}, t} \; & t \\ & Cx \leq d \\ & x^T BF \geq 0 \\ & F\lambda + t1 \leq A^T x \\ & \lambda \geq 0 \end{aligned}$$

- Only need to look at strict orders (permutations)
  - Still exponential # of LPs!

MILP formulation

- Use integer variables to encode the ordering
- z_{ij} binary integer that indicates whether adversary utility for action i is better than utility for action j

- Mixed integer quadratic program; can transform to MILP

$$
\begin{aligned}
\max_{\boldsymbol{x},\boldsymbol{w},t,\boldsymbol{z}} \quad & t \\
& C\boldsymbol{x} \leq d \\
& \boldsymbol{x}^T B \boldsymbol{e_i} + M(1 - z_{ij}) \geq \boldsymbol{x}^T B \boldsymbol{e_j}, \ \forall i, j \\
& \sum_{i,j} w_{ij}(\boldsymbol{e_i} - \boldsymbol{e_j}) + t\mathbf{1} \leq A^T \boldsymbol{x} \\
& 0 \leq w_{ij} \leq z_{ij} N \\
& z_{ij} \in \{0, 1\} \\
& z_{ij} + z_{ji} \geq 1 \\
& (1 - z_{ij}) + (1 - z_{jk}) + z_{ik} \geq 1.
\end{aligned}
$$

Top-monotonic maximin

- Top-monotonicity: the best response action is played with higher probability than other actions
  - For each action i,

$$x^T B e_i \geq x^T B e_j \; \forall j \Rightarrow y_i \geq y_j \; \forall j.$$

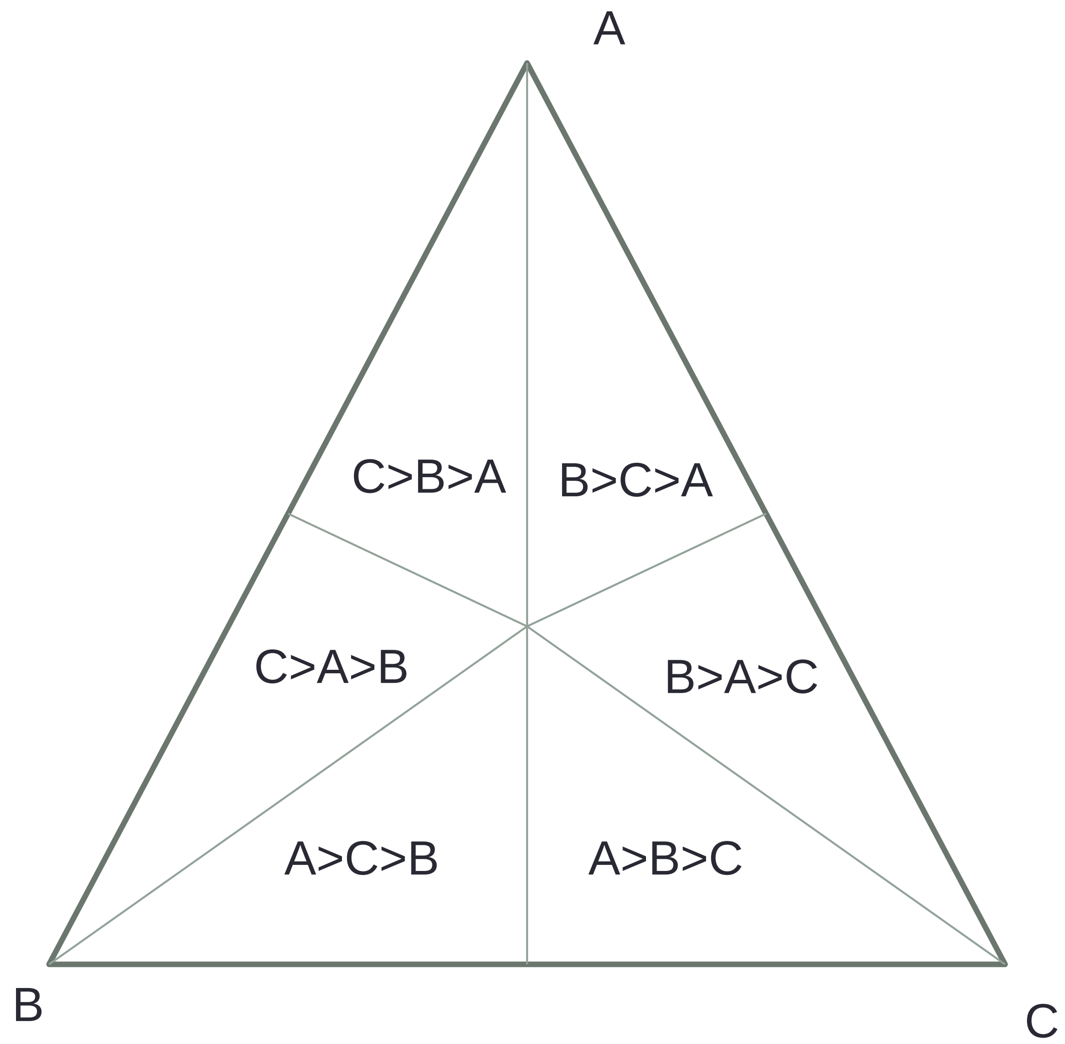
- Top-monotonic maximin: defined analogously

$$\arg\max_{x \in X} \min_{y \in \widehat{Q}(x)} x^T A y$$

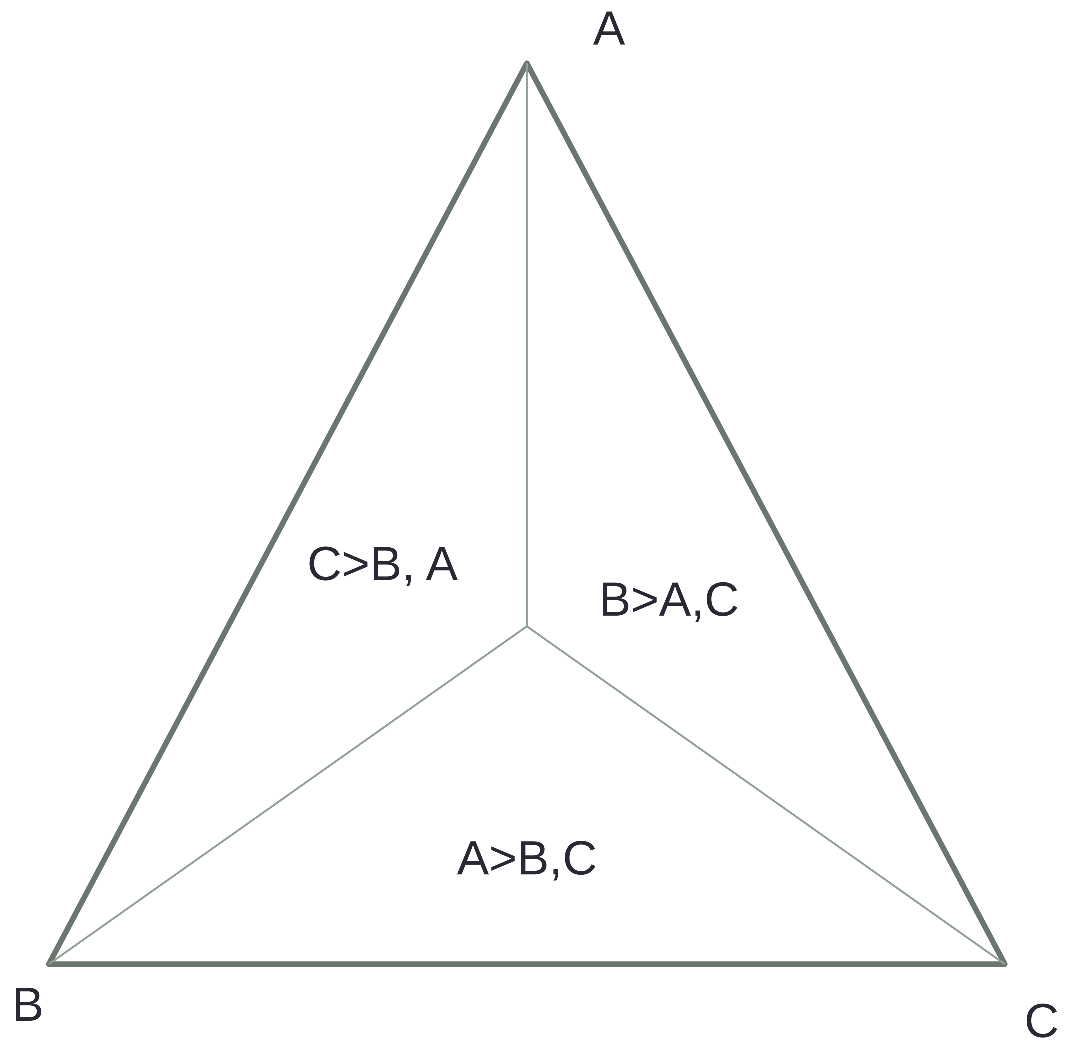
  - Lower bound on MM, i.e. more conservative

- Computation: polynomial time
  - solve n LPs, one for the case of action i being best response

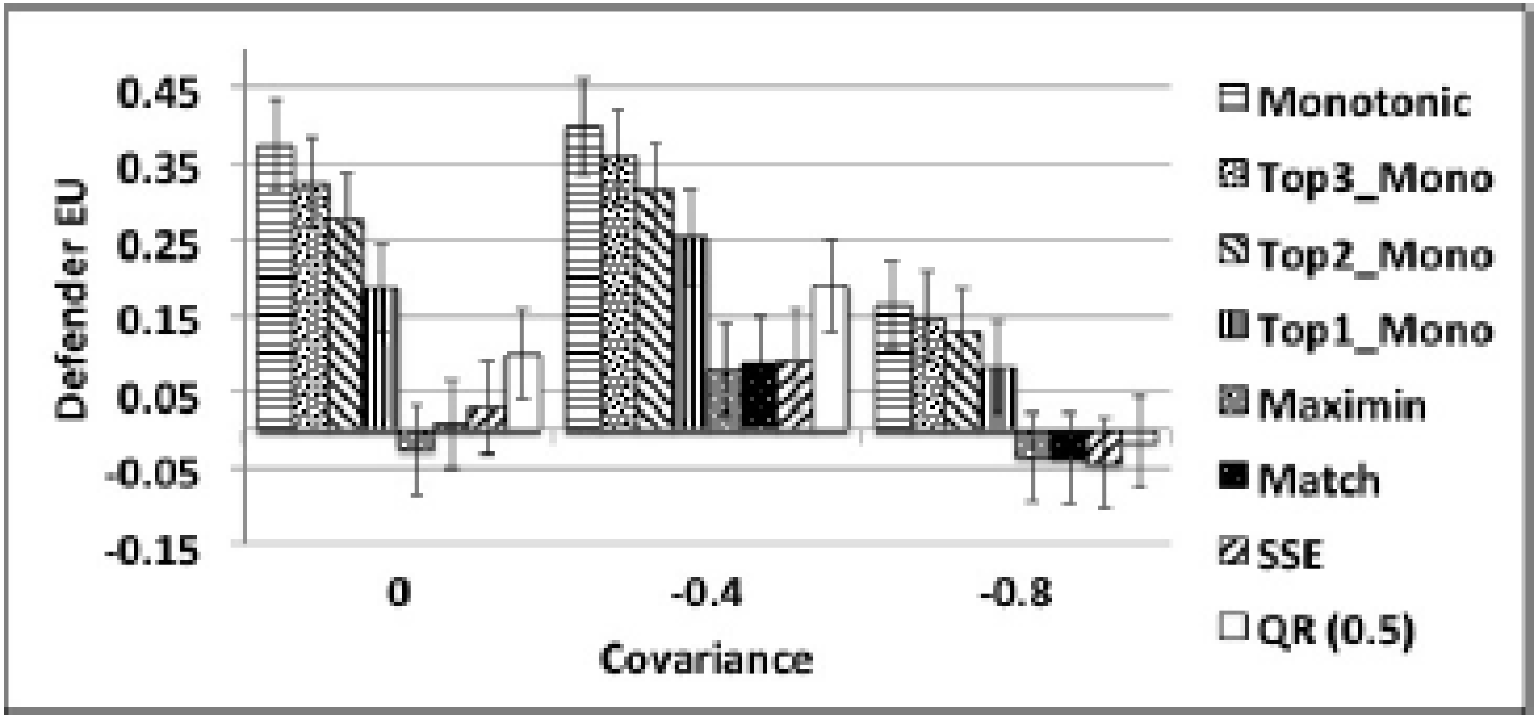
Partitioning of X: monotonic maximin



Partitioning of X: top-monotonic

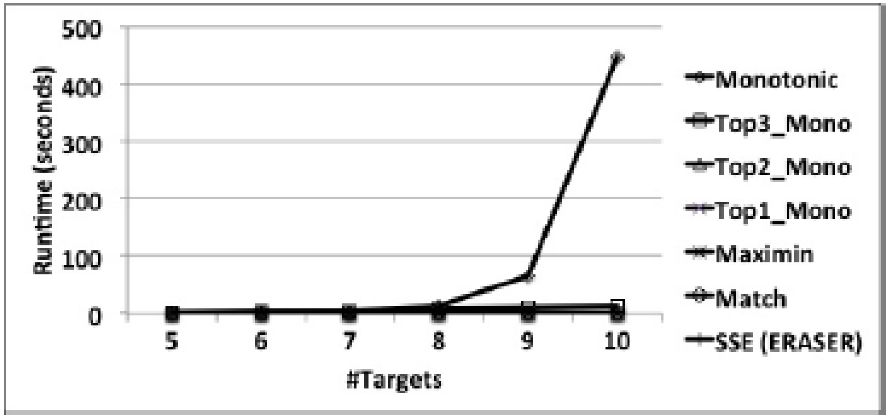


Experiments: solution quality

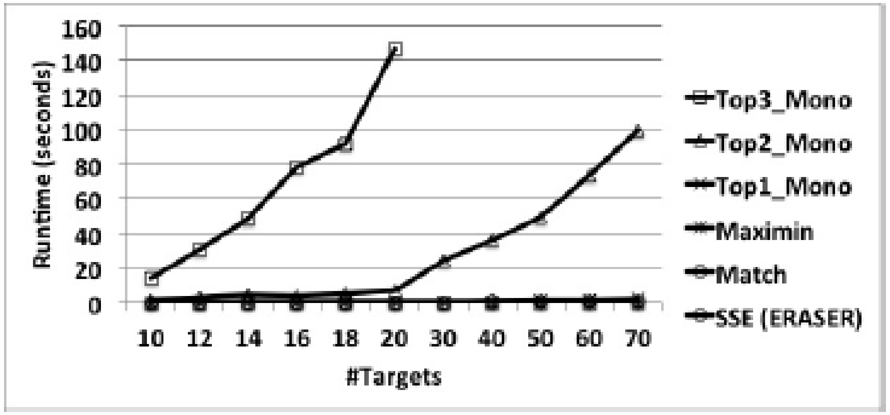


(a) 6 Targets, 3 Defender Resources

Runtime performance



(a) 5-10 Targets, 3 Defender Resources



(b) 10-70 Targets, 6 Defender Resources

Conclusions

- A robust-optimization approach to dealing with bounded rationality in Stackelberg games
  - *Monotonic maximin: robust against any monotonic adversary*
  - *Computing MM: formulate as MILP*
  - *Top-monotonic maximin: a more conservative solution; easier to compute*

Future Work and Open Problems

- More efficient computation
- Relations to / combining with other uncertainties
- How to incorporate data
- Multiple followers
  - Replacing QRE with monotonic version