HW2 keys

Question 1

a)

$$\frac{C \wedge J}{C}$$

Valid Simplification

b)

$$\frac{\neg S \rightarrow W \quad W}{\neg S}$$

Invalid Fallacy of Modus Tollens

c)

$$\frac{C \rightarrow T \quad \neg T}{\neg C}$$

Valid Modus Tollens

d)

$$\frac{P \vee K \quad \neg K}{P}$$

Valid Disjunctive Inference
(Disjunctive Syllogism)

e)

$$\frac{I \rightarrow S \quad \neg I}{\neg R}$$

Invalid Fallacy of Modus Ponens

Question 2

a)

1) $P \wedge \neg Q$
2) R

$(P \wedge R) \vee Q$

3) P	1 Simplification
4) $P \wedge R$	3, 2 Conjunctive Addition
5) $(P \wedge R) \vee Q$	4 Disjunctive Addition

b)

1) P
2) $P \rightarrow Q$
3) $\neg Q \vee R$

R

4) Q	2, 1 Modus Ponens

5) R	3, 4 Disjunctive Inference

c)

1) $P \rightarrow Q$
2) $\neg Q$
3) $\neg R$

$\neg (P \vee R)$

4) $\neg P$	1, 2 Modus Tollens
5) $\neg P \wedge \neg R$	4, 3 Conjunctive Addition
6) $\neg (P \vee R)$	5 De Morgan’s Law

d)

1) $P \rightarrow Q$
2) $R \rightarrow \neg Q$
3) R

$\neg P$

4) $\neg Q$	2, 3 Modus Ponens
5) $\neg P$	1, 4 Modus Tollens

e)

1) $P \rightarrow (Q \rightarrow R)$
2) $\neg Q \rightarrow \neg P$
3) P

R

4) $\neg (\neg Q)$	2, 3 Modus Tollens
5) Q	4 Double Negation
6) $Q \rightarrow R$	1, 3 Modus Ponens
7) R	6, 5 Modus Ponens

f)

1) $P \wedge Q$
2) $P \rightarrow (R \wedge Q)$
3) $R \rightarrow (S \vee T)$
4) $\neg S$

T

5) P	1 Simplification
6) $(R \wedge Q)$	2, 5 Modus Ponens
7) R	6 Simplification
8) $(S \vee T)$	3, 7 Modus Ponens
9) T	8, 4 Disjunctive Inference

g)

1) $P \rightarrow (Q \rightarrow R)$
2) $P \vee S$
3) $T \rightarrow Q$
4) $\underline{\neg S}$
$\neg R \rightarrow \neg T$

5) P	2, 4 Disjunctive Inference
6) $(Q \rightarrow R)$	1, 5 Modus Ponens
7) $T \rightarrow R$	3, 6 Chain Rule
8) $\neg R \rightarrow \neg T$	7 Contrapositive

h)

1) $P \vee Q$
2) $\neg P \vee R$
3) $\underline{\neg R}$
Q

4) $\neg P$	2, 3 Disjunctive Inference
5) Q	1, 4 Disjunctive Inference

Question 3

a)

1) $(S \vee H) \rightarrow R$
2) $R \rightarrow B$
3) $\underline{\neg B}$
$\neg S \vee \neg H$

Proof by Contradiction

Suppose the argument is invalid that is when all premises are true, it leads the conclusion to be false. Since we assume the argument invalid, we have to try to find at least one situation to make all premises to be true and conclusion to be false. If we can do so, then our assumption is right. (To show something wrong, if there is at least one case wrong, then the whole thing is wrong.)

[1] To make the conclusion to be False, $\neg S \vee \neg H$ must be False. So both $\neg S$ and $\neg H$ must be False. S is True. H is True.
[2] To make premise 1) to be true and S and H is True from [1] that is $S \vee H$ is True, R must be True.
[3] To make premise 2) to be true and R is True from [2], B must be True.
[4] To make premise 3) to be true, $\neg B$ must be True. B is False.

Contradiction

Since we cannot make all premises to be true and conclusion to be false without the contradiction, then our assumption was wrong.

So the argument is valid.

b)

1) $R \to H$
2) $P \to C$
3) $(H \lor C) \to N$
4) $\neg N$

$\neg R \land \neg P$

Proof by Contradiction

Suppose the argument is invalid. All premises are true. But the conclusion is false.

If we try to make the conclusion to be false now, there will be three cases to make $\land$ to be false. We don't want to start with many cases now. So, first, let so with some other premises that have only one case.

[1] To make premise 4) to be true, $\neg N$ must be True. N is False.
[2] To make premise 3) to be true and N is False from [1], $H \lor C$ must be False. So both H and C must be False.
[3] To make premise 2) to be true and C is False from [2], P must be False.
[4] To make premise 1) to be true and h is False from [2], R must be False.
[5] To make the conclusion to be False, there are three cases:

[5.1] $\neg R$ can be False. $\neg P$ can be True. So P is False. R is True.
Contradiction

[5.2] $\neg R$ can be True. $\neg P$ can be False. So R is False. P is True.
Contradiction

[5.3] $\neg R$ can be False. $\neg P$ can be False. So P is True. R is True.
Contradictions

All the cases lead us to contradiction. Another word, we cannot make all premises to be true and the conclusion to be false without contradiction. So our assumption was wrong. So the argument is valid.

c)

1) $(R \lor M) \to \neg L$
2) $O \to \neg R$
3) $O \land \neg M$

L

Proof by Contradiction

Suppose the argument is invalid. All premises are true. But the conclusion is false.

[1] To make the conclusion to be false, L must be False.

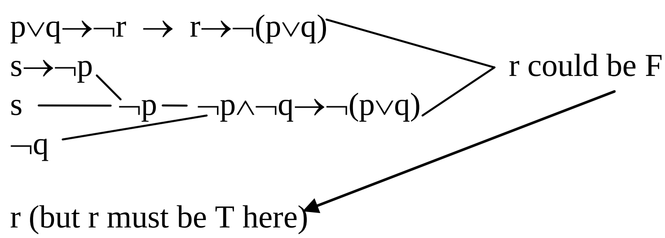
{*** We should not do the following [2]
[2] To make the premise 1) to be true and ¬L is True from [1] (L is False), (R ∨ M) can either True or False. That is four cases. So we should work with other premises first and come back for this premise later. ***}

[2] To make the premise 3) to be true, O must be True and ¬M must be true. M is False.

[3] To make the premise 2) to be true and O is true from [2], ¬R must be True. R is False.

[4] To make premise 1) to be true and M is False from [2] and R is False from [3], ¬L can be true. (False → True is True.) So L can be False.

Since we can show one situation that all premises are true and conclusion is false without a contradiction, therefore our assumption was right. The argument is invalid.



Therefore INVALID
Counterexample is : p=0, q=0, s=1, r=0

Question 4

U = {…, -3, -2, -1, 0, 1, 2, 3, … }
P(x) = {1, 2, 3, 4, …}
Q(x) = {…, 6, 4, 2, 0, 2, 4, 6 …}
R(x) = {0, 1, 4, 9, 16, …}
S(x) = {…, -12, -8, -4, 0, 4, 8, 12, …}
T(x) = {…, -15, -10, -5, 0, 5, 10, 15, …}

a)
i) At least one integer is even.
$\exists x (Q(x))$
ii) There exists a positive integer that is even.
Another word is that there is a number that is both positive and even.
$\exists x (P(x) \land Q(x))$

iii) If x is even, then x is not divisible by 5.

This case defines in general for any number in Universe. So it is for all.

$\forall x\ (\ Q(x) \rightarrow \neg T(x)\ )$

It is the same as "Even number is not divisible by 5."

iv) No even integer is divisible by 5.

For all x's, it is not true that there is a number that is even and divisible by 5.

$\forall x \neg\ (\ Q(x) \wedge T(x)\ )$

$\equiv \neg\ \exists x\ (\ Q(x) \wedge T(x)\ )$

"None of them" is equivalent to "Not even some of them".

v) There exits an even integer divisible by 5.

Another word is that there is a number that is both even integer and divisible by 5.

$\exists x\ (\ Q(x) \wedge T(x)\ )$

vi) If x is even and x is a perfect square, then x is divisible by 4.

Again this is in general for any number in Universe.

$\forall x\ (\ [\ Q(x) \wedge R(x)\ ] \rightarrow S(x)\ )$

b)

i) True because we can find at least one integer that is even such as 0, 4, -10, etc.

ii) True because we can find at least one integer that is both positive and even such as 2, 4, 6, etc.

iii) False because 10, 20 are even but are divisible by 5.

(10, 20 are counterexamples.) (Counterexample is an example to challenge that the statement doesn't have to be true.)

So it is not true that if x is even, then x is not divisible by 5.

≡ It is not true for all number that if it is even, then it is not divisible by 5.

iv) False because there is some even integer that is divisible by 5 such as 20, 30.

(20, 30 are counterexample to show that the statement doesn't have to be true.)

Actually iii) is equivalent to iv)

iii) $\forall x\ (\ Q(x) \rightarrow \neg T(x)\ )$

$\equiv \forall x\ (\ \neg Q(x) \vee \neg T(x)\ )$

$\equiv \forall x\ \neg(\ Q(x) \wedge T(x)\ )$

$\equiv \neg\ \exists x\ (\ Q(x) \wedge T(x)\ )$ iv)

v) True because there is some even integer that is divisible by 5 such as 10, 20, etc.

vi) True because all even perfect squares must be at least a factor of 2*2.
To be a square, the number must be a multiple of 2 identical numbers. Therefore an even perfect square must be (at least) a factor of 2*2.
We cannot find a counterexample to show an even perfect square that is not divisible by 4.

c)

i) $\forall x$ [R(x) $\rightarrow$ P(x)]
For all integers x if x is a perfect square then x is greater than 0.
$\equiv$ A perfect square is greater than 0.
$\equiv$ A perfect square is nonnegative.

ii) $\forall x$ [S(x) $\rightarrow$ Q(x)]
For all integers x if x is divisible by 4 then x is even.
$\equiv$ An integer that is divisible by 4 is even.
!$\equiv$ An integer that is even is divisible by 4. They are not equivalent.

iii) $\forall x$ [S(x) $\rightarrow \neg$T(x)]
For all integers x if it is divisible by 4 then it is not divisible by 5.
$\equiv$ An integer that is divisible 4 is not divisible by 5.

iv) $\exists x$ [S(x) $\wedge \neg$R(x)]
There is some integer that is divisible by 4 but not a perfect square.

d)

i) True because the square of negative, zero, and positive are always positive.

ii) True because when a number is divisible by 4, that number is also divisible by 2. And a number is divisible by 2 if and only if it is even.

iii) False. The counterexamples are 20, 40, … etc.
Actually we don't need to show all counterexample. Only one example is good enough to say False.

iv) True such as 8, 12, 20, 24, 28, etc.
But 4, 16, 36, 64, … cannot be such examples.

Question 5

a) $\neg \exists x$[P(x) $\vee$ Q(x)]
Laws of the negation of a Quantifier

$\equiv \forall x \neg[P(x) \vee Q(x)]$

De Morgan's Law

$\equiv \forall x [\neg P(x) \wedge \neg Q(x)]$

b) $\neg \forall x [P(x) \wedge \neg Q(x)]$

Laws of the negation of a Quantifier

$\equiv \exists x \neg[P(x) \wedge \neg Q(x)]$

De Morgan's Law

$\equiv \exists x [\neg P(x) \vee Q(x)]$

c) $\neg \forall x [P(x) \rightarrow Q(x)]$

Laws of the negation of a Quantifier

$\equiv \exists x \neg[P(x) \rightarrow Q(x)]$

Truth of $\rightarrow$

$\equiv \exists x \neg[\neg P(x) \vee Q(x)]$

De Morgan's Law

$\equiv \exists x [P(x) \wedge \neg Q(x)]$

d) $\neg \exists x [(P(x) \vee Q(x)) \rightarrow R(x)]$

Laws of the negation of a Quantifier

$\equiv \forall x \neg[(P(x) \vee Q(x)) \rightarrow R(x)]$

Truth of $\rightarrow$

$\equiv \forall x \neg[\neg(P(x) \vee Q(x)) \vee R(x)]$

De Morgan's Law

$\equiv \forall x [(P(x) \vee Q(x)) \wedge \neg R(x)]$

Distributive Law (if you'd like to)

$\equiv \forall x [(P(x) \wedge \neg R(x)) \vee (Q(x)) \wedge \neg R(x))]$