HW3 Mathematic Induction and Recursive Definition
(Due at the beginning of class Wed. Feb. 23)

Use your own paper and show work as much as possible.

Mathematic Induction

1. (5 points each, total 20 points) Prove each of the following for all $n \in Z^+$ by the Principle of Mathematical Induction.
   a. $1^2+3^2+5^2+ \bullet\bullet\bullet +(2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$
   b. $1\cdot 3+2\cdot 4+3\cdot 5+\cdots+n(n+2)=\dfrac{n(n+1)(2n+7)}{6}$
   c. $\sum_{i=1}^{n} \dfrac{1}{i(i+1)} = \dfrac{n}{n+1}$
   d. $\sum_{i=1}^{n} i^3 = \dfrac{n^2(n+1)^2}{4}$

There are two parts to proof by Mathematical Induction
    1) Basic Step: proof s(1) (1 means the first value of s(n).)
    2) Inductive Step: proof s(k+1) by assuming that s(n) is true from 1 to k.

For each step, we have to show whether the left side of "=" equals to the right side of "=". For each step of each problem of question 1, we have to show whether the summation equals to the formula.

a) $s(n) = 1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \sum_{i=1}^{n} (2i-1)^2 = (n)(2n-1)(2n+1)/3$

Basic Step: n = 1

$$
\begin{aligned}
s(1) = (n)(2n-1)(2n+1)/3 &= [1][2(1) - 1][2(1) + 1] / 3 \\
&= [1][1][3] / 3 \\
&= 1 \qquad (1)
\end{aligned}
$$

$$\sum_{i=1}^{1} (2i-1)^2 = 1^2 = 1 \qquad (2)$$

◆ (1) = (2). So the summation equals to the formula in the basic step.

Inductive Step: n = k+1

Assume that s(k) is true.

$$\begin{aligned} s(k+1) &= [k+1][2(k+1) - 1][2(k+1) + 1] / 3 \\ &= [k+1][2k+2 - 1][2k+2 + 1] / 3 \\ &= [k+1][2k+1][2k+3] / 3 \qquad (3) \end{aligned}$$

For $\sum_{i=1}^{k+1} (2i-1)^2$, since we assume that s(k) is true, we split them into two parts, summation of $(2i-1)^2$ from i = 1 to k and another part is when i=k+1.

$$\begin{aligned} \sum_{i=1}^{k+1} (2i-1)^2 &= 1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2 + (2(k+1) - 1)^2 \\ &= 1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2 + (2k+2 - 1)^2 \\ &= 1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2 + (2k+1)^2 \\ &= \sum_{i=1}^{k} (2i-1) + (2k+1)^2 \end{aligned}$$

Since we do believe that s(k) is true, we can use the formula for i=1 to k

$$\begin{aligned} \sum_{i=1}^{k+1} (2i-1)^2 &= \sum_{i=1}^{k} (2i-1) + (2k+1)^2 \\ &= [k][2k-1][2k+1]/3 + (2k+1)^2 \\ &= (2k+1) \left([k][2k-1]/3 + [2k+1] \right) \\ &= (2k+1) \left([k][2k-1] + 3[2k+1] \right) / 3 \\ &= (2k+1) \left(2k^2 - k + 6k+3 \right) / 3 \\ &= (2k+1) \left(2k^2 + 5k + 3 \right) / 3 \\ &= (2k+1) (2k + 3)(k + 1) / 3 \\ &= (k+1)(2k+1)(2k+3)/3 \qquad (4) \end{aligned}$$

- (3) = (4). So the summation equals to the formula in the inductive step.

It is true for both basic step, s(1), and inductive step, when s(k) → s(k+1).

Therefore $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \sum_{i=1}^{n} (2i - 1)^2 = (n)(2n-1)(2n+1)/3$.

b) $1.3 + 2.4 + 3.5 + \ldots + n(n+2) = (n)(n+1)(2n+7)/6$

$$= \sum_{i=1}^{n} i(i+2)$$

Basic Step: n = 1

$$s(1) = (n)(n+1)(2n+7)/6 = [1][1 + 1][2(1) + 7] / 6$$
$$= [1][2][9] / 6$$
$$= 3 \qquad (1)$$

$$\sum_{i=1}^{1} i(i + 2) = 1(1+2) = (1)(3) = 3 \qquad (2)$$

♦ (1) = (2). So the summation equals to the formula in the basic step.

Inductive Step: n = k+1

Assume that s(k) is true.

$$s(k+1) = [k+1][(k+1) + 1][2(k+1) + 7] / 6$$
$$= [k+1][k+2][2k+2 + 7] / 6$$
$$= [k+1][k+2][2k+9] / 6 \qquad (3)$$

For $\sum_{i=1}^{k+1} i(i +2)$, since we assume that s(k) is true, we split them into two parts, summation of i(i + 2) from i = 1 to k and another part is when i=k+1.

$$\sum_{i=1}^{k+1} i(i + 2) = 1.3 + 2.4 + 3.5 + \ldots + k(k +2) + (k+1)(k+1 + 2)$$
$$= 1.3 + 2.4 + 3.5 + \ldots + k(k+2) + (k+1)(k+3)$$
$$= \sum^{k} i(i + 2) + (k+1)(k+3)$$

$i=1$

Since we do believe that s(k) is true, we can use the formula for i=1 to k

$$\begin{aligned}
\sum_{i=1}^{k+1} i(i+2) &= \sum_{i=1}^{k} i(i+2) + (k+1)(k+3) \\
&= [k][k+1][2k+7]/6 + (k+1)(k+3) \\
&= (k+1)\ (\ [k][2k+7]/6 + [k+3]\ ) \\
&= (k+1)\ (\ [k][2k+7] + 6[k+3]\ )/6 \\
&= (k+1)\ (\ 2k^2 + 7k + 6k+18\ )/6 \\
&= (k+1)\ (\ 2k^2 + 13k + 18\ )/6 \\
&= (k+1)\ (2k+9)(k+2)/6 \\
&= (k+1)(k+2)(2k+9)/6 \qquad (4)
\end{aligned}$$

- (3) = (4). So the summation equals to the formula in the inductive step.

It is true for both basic step, s(1), and inductive step, when s(k) → s(k+1).

Therefore $1.3 + 2.4 + 3.5 + \ldots + i(i+2) = \sum_{i=1}^{n} i(i+2) = (n)(n+1)(2n+7)/6$.

c) $s(n) = \sum_{i=1}^{n} 1/[i(i+1)] = (n)/(n+1)$

Basic Step: n = 1

$$\begin{aligned}
s(1) = (n)/(n+1) &= [1]/[1+1] \\
&= 1/2 \qquad (1)
\end{aligned}$$

$$\sum_{i=1}^{1} 1/[1\,(1+1)] = ½ \qquad (2)$$

◆ (1) = (2). So the summation equals to the formula in the basic step.

Inductive Step: n = k+1

Assume that s(k) is true.

$$s(k+1) = [k+1] / [(k+1) + 1]$$
$$= [k+1] / [k + 2] \qquad (3)$$

For $\sum_{i=1}^{k+1} 1/ [i (i + 1)]$, since we assume that s(k) is true, we split them into two parts, summation from i = 1 to k and another part is when i=k+1.

$$\sum_{i=1}^{k+1} 1 /[i (i +1)] = \sum_{i=1}^{k} 1/[i (i+1)] + 1 / [(k+1)(k+1+1)]$$

$$= \sum_{i=1}^{k} 1/[i (i+1)] + 1 / [(k+1)(k+2)]$$

Since we do believe that s(k) is true, we can use the formula for i=1 to k

$$\sum_{i=1}^{k+1} 1 /[i (i +1)] = \sum_{i=1}^{k} 1/[i (i+1)] + 1 / [(k+1)(k+2)]$$

$$= [k] / [k+1] + 1 / [(k+1)(k+2)]$$

$$= [k + 1/(k+2)] / [k+1]$$

$$= [(k(k+2) + 1) / (k+2)] / [k+1]$$

$$= [(k^2+2k + 1) / (k+2)] / [k+1]$$

$$= [k^2+2k + 1] / [k+2][k+1]$$

$$= (k+1)^2 / [k+2][k+1]$$

$$= (k+1) / (k+2) \qquad (4)$$

◆ (3) = (4). So the summation equals to the formula in the inductive step.

It is true for both basic step, s(1), and inductive step, when s(k) → s(k+1).

Therefore $= \sum_{i=1}^{n} 1 / [i(i+1)] = (n)/(n+1)$.

d) $s(n) = \sum_{i=1}^{n} i^3 = (n^2)(n+1)^2/4 = (\sum_{i=1}^{n} i)^2$

Basic Step: n = 1

$$s(1) = (1^2)(1+1)^2/4 = [1][2]^2 / 4 = 4 / 4 = 1 \quad (1)$$

$$\sum_{i=1}^{1} i^3 = 1^3 = 1 \quad (2)$$

$$(\sum_{i=1}^{1} i)^2 = (1)^2 = 1 \quad (3)$$

- (1) = (2) = (3). So the summation equals to the formula in the basic step.

Inductive Step: n = k+1

Assume that s(k) is true.

$$s(k+1) = [k+1]^2[(k+1) + 1]^2 / 4$$
$$= [k+1]^2[k+2]^2 / 4 \quad (4)$$

For $\sum_{i=1}^{k+1} i^3$, since we assume that s(k) is true, we split them into two parts, summation of i^3 from i = 1 to k and another part is when i=k+1.

$$\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3$$

Since we do believe that s(k) is true, we can use the formula for i=1 to k

$$\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3$$

$$= [k]^2[k+1]^2 / 4 + (k+1)^3$$

$$= (k+1)^2 \ ([k]^2/4 + [k+1])$$

$$= (k+1)^2 \ ([k]^2 + 4[k+1]) / 4$$

$$= (k+1)^2 \ (k^2 + 4k + 4) / 4$$

$$= (k+1)^2 \ (k + 2)^2 / 4 \qquad (5)$$

For $(\sum_{i=1}^{k+1} i)^2$, since we assume that s(k) is true, we split them into two parts, summation from i = 1 to k and another part is when i=k+1.

$$(\sum_{i=1}^{k+1} i)^2 = (\sum_{i=1}^{k} i + (k+1))^2$$

$\{ (a + b)^2 = a^2 + 2ab + b^2 \}$

$$= (\sum_{i=1}^{k} i)^2 + 2 (\sum_{i=1}^{k} i) (k+1) + (k+1)^2$$

Since we do believe that s(k) is true, we can use the formula for i=1 to k

$$(\sum_{i=1}^{k+1} i)^2 = (\sum_{i=1}^{k} i)^2 + 2 (\sum_{i=1}^{k} i) (k+1) + (k+1)^2$$

$$= [k]^2[k+1]^2 / 4 + 2 (\sum_{i=1}^{k} i) (k+1) + (k+1)^2$$

$\{ \sum_{i=1}^{n} i = n(n+1)/2 \}$

$$= \quad [k]^2[k+1]^2 / 4 \ + \ 2\,(k(k+1)/2)\,(k+1) \ + \ (k+1)^2$$

$$= \quad k^2(k+1)^2 / 4 \ + \ k(k+1)\,(k+1) \ + \ (k+1)^2$$

$$= \quad (k)^2(k+1)^2 / 4 \ + \ k(k+1)^2 \ + \ (k+1)^2$$

$$= \quad (k+1)^2 \ [\ (k)^2 / 4 \ + \ k \ + \ 1\ ]$$

$$= \quad (k+1)^2 \ [\ (k)^2 \ + \ 4k \ + \ 4]\ /\ 4$$

$$= \quad (k+1)^2\,(k+2)^2 /\ 4 \qquad (6)$$

♦ (4) = (5) = (6). So the summation equals to the formula in the inductive step.

It is true for both basic step, s(1), and inductive step, when s(k) → s(k+1).

Therefore $\sum_{i=1}^{n} i^3 \quad = \quad (n^2)(n+1)^2/4 \quad = \quad (\sum_{i=1}^{n} i\,)^2$

2. (6 points each, total 18 points) Establish each of the following for all $n \in Z^+$ by the Principle of Mathematical Induction.
    a. $\sum_{i=1}^{n} 2^{i-1} = \sum_{i=0}^{n-1} 2^i = 2^n - 1$
    b. $\sum_{i=1}^{n} i(2^i) = 2 + (n-1)2^{n+1}$
    c. $\sum_{i=1}^{n} (1)(i!) = (n+1)! - 1$

a) $s(n) = \sum_{i=1} 2^{i-1} \quad = \quad \sum_{i=0} 2^i \quad = \quad 2^n - 1$

$\sum_{i=1}^{n} 2^{i-1} \quad = \quad 2^n - 1$

Basic Step: n = 1

$$s(1) \quad = \ 2^1 - 1 \qquad = \quad 2 - 1$$
$$= \quad 1 \qquad (1)$$

$$\sum_{i=1}^{1} 2^{i-1} = 2^{1-1} = 2^{0} = 1 \qquad (2)$$

$$\sum_{i=0}^{1-1} 2^{i} = \sum_{i=0}^{0} 2^{i} = 2^{0} = 1 \qquad (3)$$

- ♦ (1) = (2) = (3). So the summation equals to the formula in the basic step.

Inductive Step: n = k+1

Assume that s(k) is true.

$$s(k+1) = 2^{k+1} - 1 \qquad (4)$$

For $\sum_{i=1}^{k+1} 2^{i-1}$, since we assume that s(k) is true, we split them into two parts, summation of 2^{i-1} from i = 1 to k and another part is when i=k+1.

$$\sum_{i=1}^{k+1} 2^{i-1} = 2^{1-1} + 2^{2-1} + 2^{3-1} + \ldots + 2^{k-1} + 2^{k+1-1}$$

$$= \sum_{i=1}^{k} 2^{i-1} + 2^{k}$$

Since we do believe that s(k) is true, we can use the formula for i=1 to k

$$\sum_{i=1}^{k+1} 2^{i-1} = 2^{k} - 1 + 2^{k}$$

$$= 2^{k} + 2^{k} - 1$$

$\{ a + a = 2a \}$

$$= (2)(2^{k}) - 1$$

$\{a^{n} . a^{m} = a^{n+m} \}$

$$= \quad 2^{k+1} - 1 \qquad (5)$$

For $\sum_{i=0}^{k+1-1} 2^i = \sum_{i=0}^{k} 2^i$, we assume that $s(k) = \sum_{i=0}^{k-1} 2^i$ is true.

$$\sum_{i=0}^{k+1-1} i\ 2^i = \sum_{i=0}^{k} 2^i$$

$$= \sum_{i=0}^{k-1} 2^i + 2^k$$

Since we do believe that s(k) is true, we can use the formula for s(k).

$$= 2^k - 1 + 2^k$$

$$= 2(2^k) - 1$$

$$= 2^{k+1} - 1 \qquad (6)$$

♦ (4) = (5) = (6). So the summation equals to the formula in the inductive step.

It is true for both basic step, s(1), and inductive step, when s(k) → s(k+1).

Therefore $\sum_{i=1}^{n} 2^{i-1} = \sum_{i=0}^{n-1} 2^i = 2^n - 1$

b) $s(n) = \sum_{i=1}^{n} i(2^i) = 2 + (n-1)2^{n+1}$

Basic Step: n = 1

$$s(1) = 2 + (1-1)\,2^{1+1} = 2 + (0)\,2^0$$
$$= 2 \qquad (1)$$

$$\sum_{i=1}^{1} i(2^i) = 1(2^1) = 2 \qquad (2)$$

♦ (1) = (2). So the summation equals to the formula in the basic step.

Inductive Step: $n = k+1$

Assume that $s(k)$ is true.

$$\begin{aligned} s(k+1) &= 2 + (k+1-1)2^{k+1+1} \\ &= 2 + (k)2^{k+2} \qquad (3) \end{aligned}$$

For $\sum_{i=1}^{k+1} i(2^i)$, since we assume that $s(k)$ is true, we split them into two parts, summation of $i(2^i)$ from $i = 1$ to k and another part is when $i=k+1$.

$$\sum_{i=1}^{k+1} i(2^i) = \sum_{i=1}^{k} i(2^i) + (k+1)(2^{k+1})$$

Since we do believe that $s(k)$ is true, we can use the formula for $i=1$ to k

$$\begin{aligned} \sum_{i=1}^{k+1} i(2^i) &= 2 + (k-1)2^{k+1} + (k+1)(2^{k+1}) \\ &= 2 + (2^{k+1})[(k-1) + (k+1)] \\ &= 2 + (2^{k+1})[2k] \\ &= 2 + k(2^{k+1})2 \\ &= 2 + k(2^{k+1+1}) \\ &= 2 + k(2^{k+2}) \qquad (4) \end{aligned}$$

♦ (3) = (4). So the summation equals to the formula in the inductive step.

It is true for both basic step, $s(1)$, and inductive step, when $s(k) \rightarrow s(k+1)$.

Therefore $\sum_{i=1}^{n} i(2^i) = 2 + (n-1)2^{n+1}$

c) $s(n) = \sum_{i=1}^{n} (i)(i!) = (n+1)! - 1$

Basic Step: n = 1

$$s(1) = (1+1)! - 1 = 2! - 1 = 2 - 1 = 1 \qquad (1)$$

$$\sum_{i=1}^{1} i(i!) = 1(1!) = 1 \qquad (2)$$

◆ (1) = (2). So the summation equals to the formula in the basic step.

Inductive Step: n = k+1

Assume that s(k) is true.

$$s(k+1) = (k+1+1)! - 1 = (k+2)! - 1 \qquad (3)$$

For $\sum_{i=1}^{k+1} i(i!)$, since we assume that s(k) is true, we split them into two parts, summation of i(i!) from i = 1 to k and another part is when i=k+1.

$$\sum_{i=1}^{k+1} i(i!) = \sum_{i=1}^{k} i(i!) + (k+1)(k+1)!$$

Since we do believe that s(k) is true, we can use the formula for i=1 to k

$$\sum_{i=1}^{k+1} i(i!) = (k+1)! - 1 + (k+1)(k+1)!$$

$$= (k+1)! + (k+1)(k+1)! - 1$$

$$= (k+1)! [1 + (k+1)] - 1$$

$$= (k+1)! [k+2] - 1$$

$\{ (n+2)! = (n+2)(n+1)! \}$

$$= (k+2)! - 1 \qquad (4)$$

◆ (3) = (4). So the summation equals to the formula in the inductive step.

It is true for both basic step, s(1), and inductive step, when s(k) → s(k+1).

Therefore $\sum_{i=1}^{n} (i)(i!) = (n+1)! - 1$

3. (8 points) Determine the positive integer n for which

$$\sum_{i=1}^{2n} i = \sum_{i=1}^{n} i^2$$

$\sum_{i=1}^{n} i = n(n+1)/2$

$\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$

$$\sum_{i=1}^{2n} i = \sum_{i=1}^{n} i^2$$

$$2n(2n + 1)/2 = n(n+1)(2n+1)/6$$

$$n(2n + 1) = n(n+1)(2n+1)/6$$

$$(2n + 1) = (n+1)(2n+1)/6$$

$$1 = (n+1)/6$$

$$6 = (n+1)$$

$n = 5$ Answer.

4. (8 points) Consider the following four equations:

a. $1 = 1$

b. $2+3+4 = 1+8$

c. $5+6+7+8+9 = 8+27$

d. $10+11+12+13+14+15+16 = 27+64$

Conjecture the general formula suggested by these four equations, and prove your conjecture.

$$\begin{array}{rrcl}
n=1: & 1 & = & 1 \\
2: & 2 + 3 + 4 & = & 1 + 8 \\
3: & 5 + 6 + 7 + 8 + 9 & = & 8 + 27 \\
4: & 10 + 11 + 12 + 13 + 14 + 15 + 16 & = & 27 + 64 \\
 & \dots & = & \dots \\
 & & & \\
n: & ((n-1)^2 + 1) + ((n-1)^2 + 2) + \dots + n^2 & = & (n-1)^3 + n^3
\end{array}$$

Therefore we can define it as

$$s(n) = \sum_{i=(n-1)^2 + 1}^{n^2} i = (n-1)^3 + n^3 \qquad \text{when } n >= 1$$

Proof by Mathematical Induction

Basic Step: $n = 1$

$$s(n) = (1-1)^3 + 1^3 = 0^3 + 1 = 1 \qquad (1)$$

$$\sum_{i=(n-1)^2 + 1}^{n^2} i = \sum_{i=(1-1)^2 +1}^{1^2} i = \sum_{i=1}^{1} i = 1 \qquad (2)$$

- (1) = (2). So the summation equals to the formula in the basic step.

Inductive Step: $n = k+1$ (by assuming that $s(k)$ is true.)

$$S(k+1) = (k+1 -1)^3 + (k+1)^3$$

$$= k^3 + (k+1)^3 \qquad (3)$$

$$\sum_{i=(n-1)^2 + 1}^{n^2} i = \sum_{i=(k+1-1)^2 +1}^{(k+1)^2} i$$

$$= \sum^{(k+1)^2} i \qquad (4)$$

$i=k^2 + 1$

By our assumption is, $s(k) = \sum_{i=(k-1)^2+1}^{k^2} i = (k-1)^3 + k^3$ is true when $k > 1$.

So we will split the summation to contain the form of s(k).

From (4)

$$S(k+1) = \sum_{i=k^2+1}^{(k+1)^2} i$$

$$= \sum_{i=(k-1)^2+1}^{(k+1)^2} i - \sum_{i=(k-1)^2+1}^{k^2} i$$

$$= \sum_{i=(k-1)^2+1}^{(k+1)^2} i - [(k-1)^3 + k^3] \quad \text{(because of assumption.)}$$

$$= \sum_{i=1}^{(k+1)^2} i - \sum_{i=1}^{(k-1)^2} i - [(k-1)^3 + k^3]$$

Use the formation of summation from i is 1 to n = n(n+1)/2.

$$= ((k+1)^2)((k+1)^2 +1) / 2 - ((k-1)^2)((k-1)^2 +1) / 2 - [(k-1)^3 + k^3]$$

$$= (k^2 + 2k + 1)(k^2 + 2k + 1 + 1)/2 - (k^2 - 2k + 1)(k^2 - 2k + 1 + 1)/2 - [(k-1)^3 + k^3]$$

$$= (k^2 + 2k + 1)(k^2 + 2k + 2)/2 - (k^2 - 2k + 1)(k^2 - 2k + 2)/2 - [(k-1)^3 + k^3]$$

$$= (k^4 + 2k^3 + 2k^2 + 2k^3 + 4k^2 + 4k + k^2 + 2k + 2) / 2 - (k^4 - 2k^3 + 2k^2 - 2k^3 + 4k^2 - 4k + k^2 - 2k + 2) / 2$$

$$- \ [(k-1)^3 + k^3]$$

$$\begin{aligned} = \ & (k^4 + 2k^3 + 2k^2 + 2k^3 + 4k^2 + 4k + k^2 + 2k + 2 \\ & \quad - \ k^4 + 2k^3 - 2k^2 + 2k^3 - 4k^2 + 4k - k^2 + 2k - 2) / 2 \\ & \quad - \ [(k-1)^3 + k^3] \end{aligned}$$

$$= \quad (8k^3 + 12k)/2 \qquad - \ [(k-1)^3 + k^3]$$

$$= \quad 4k^3 + 6k \qquad - \ [(k^3 - 3k^2 + 3k - 1) + k^3]$$

$$= \quad 4k^3 + 6k \qquad - \ k^3 + 3k^2 - 3k + 1 - k^3$$

$$= \quad k^3 + 3k^2 + 3k + 1 \quad + k^3$$

$$= \quad (k + 1)^3 \qquad + k^3 \qquad\qquad (5)$$

So (3) = (5).
Thus it is true for the inductive step ($s(k) \rightarrow s(k+1)$).

It is true for both basic and inductive step.

$$\text{Therefore } s(n) \quad = \quad \sum_{i=(n-1)^2 + 1}^{n^2} i \quad = \quad (n-1)^3 + n^3 \qquad \text{when } n >= 1$$

5. (8 points) During the execution of a certain program segment (written in pseudocode), the user assigns to the integer variables x and n any (possibly different) positive integers. The segment shown in the following figure immediately follows these assignments. If the program reaches the top of the **while** loop, state and prove (by mathematical induction) what the value assigned to *answer* will be after the two loop instructions are executed n(>0) times.

```
while n≠0
{
        x :=x*n
        n :=n-1
}
answer :=x
```

For x, n as read-in positive integers from a user,

```
While n ≠ 0 do
begin
        x := x * n
        n := n-1
end
answer := x
```

The segment of program calculate the multiplication of x and n that decrease by 1 from n to 1. So the segment of program should calculate x(n!). Let use the Mathematical Induction technique to proof it.

The proof is between the final answer of the segment of the program and our hypothesis, x(n!).

Basic Step: n = 1

Our hypothesis is x(n!) = x(1!) = x.

Before the execution of the loop, x is x, and n is 1. At the first loop, the condition, n ≠ 0, is true. So x is x(1) = x. Then n = 1-1=0. So the condition is false. The loop stops. Therefore the answer is x.

So it is true that the program segment calculate x(n!) for the basic step.

Inductive Step: n = k+1

From our hypothesis, the program segment should calculate x(n!) = x.[(k+1)!]

For the inductive step, we assume that the program segment calculate x(n!) until n = k. That means, if the program segment run for k loops, the program segment will calculate x(k!).

When n = k+1, there is one more loop on the first loop before k loops execution. So the final value of variable x will be x(k!) * (k+1) = x[(k+1)!].

So it is true on the inductive step.

Because our hypothesis is true for both basic and inductive step, therefore the program segment calculate x(n!).

Recursive Definition

6. (3 points each, total 18 points) The integer sequence $a_1, a_2, a_3, \ldots,$ defined explicitly by the formula $a_n=5n$ for $n \in Z^+$, can also be defined recursively by

   1) $a_1=5$; and
   2) $a_{n+1}=a_n+5$, for $n \geq 1$

   For the integer sequence $b_1, b_2, b_3, \ldots,$ where $b_n=n(n+2)$ for all $n \in Z^+$, we can also provide the recursive definition:

   1)' $b_1=3$; and
   2)' $b_{n+1}=b_n+2n+3$, for $n \geq 1$

   Give a recursive definition for each of the following integer sequences $c_1, c_2, c_3, \ldots,$ where for all $n \in Z^+$ we have

   a) $C_n=7n$
   b) $C_n=7^n$
   c) $C_n=3n+7$
   d) $C_n=7$
   e) $C_n=n^2$
   f) $C_n=2-(-1)^n$

$n \in Z^+$

To recursively define, we have to define:

1. the recursive part that define the value of the index n from the index n-1 or n-2 or n-3 or … of which index is less than n,
2. the initial-value part that is enough for any value of index n in the recursive part. Mostly we need only C_1. For some cases, we might need C_2, C_3, or C_4 or … depending on how we define the recursive part.

a) $C_n = 7n$

$$\begin{array}{lllllllll}
C_1 & = & 7(1) & = & 7 & & & & \\
C_2 & = & 7(2) & = & 14 & = & 7+7 & = & C_1+7 \\
C_3 & = & 7(3) & = & 21 & = & 14+7 & = & C_2+7 \\
C_4 & = & 7(4) & = & 28 & = & 21+7 & = & C_3+7 \\
\ldots & & & & & & & & \\
C_n & = & & & & & & & C_{n-1}+7
\end{array}$$

So the recursive part is $C_n = C_{n-1} + 7$.

If we want to know C_n, we need to know C_{n-1}.
If we want to know C_{n-1}, we need to know C_{n-2}.
…
If we want to know C_4, we need to know C_3.

If we want to know C_3, we need to know C_2.
If we want to know C_2, we need to know C_1.
(but $n \in Z^+$, no C_0)

If we want to know any C_n, we need C_1.
Another word we don't know C_1, we can't know the value of any C_n.

So we have to define C_1 for the initial-value part.
$C_1 = 7$

Therefore the recursive definition is:

$C_1 = 7$
$C_n = C_{n-1} + 7$.

b) $C_n = 7^n$

$C_1 = 7^1 = 7$
$C_2 = 7^2 = 7^1 * 7 = C1 * 7$
$C_3 = 7^3 = 7^2 * 7 = C_2 * 7$
$C_4 = 7^4 = 7^3 * 7 = C_3 * 7$
…
$C_n = 7^n = 7^{(n-1)} * 7 = C_{n-1} * 7$

So the recursive part $C_n = C_{n-1} * 7$

If we want to know a value of C at an index, n, we need to know the value of C at its previous index. So if we define C_1, we will know C_2, then C_3, then C_4 and so on. So the initial-value part must define C_1

Therefore the recursive definition is

$C_1 = 7$.
$C_n = C_{n-1} * 7$.

c) $C_n = 3n + 7$

$C_1 = 3(1) + 7 = 10$
$C_2 = 3(2) + 7 = 13$
$C_3 = 3(3) + 7 = 16$
$C_4 = 3(4) + 7 = 19$
…
$C_{n-1} = 3(n-1) + 7$
$C_n = 3(n) + 7$

The deferent value between two consecutive C is by 3.

So $C_n = C_{n-1} + 3$.

Again only one one-step previous value of C needs to get the value of C.
If we knows C_1, we can fine C_2.
If we knows C_2, we can fine C_3.
…
and so on.

So the initial part is C_1.

Therefore the recursive definition is
$C_1 = 10$
$C_n = C_{n-1} + 3$.

d) $C_n = 7$

$C_1 = 7$
$C_2 = 7$
$C_3 = 7$
$C_4 = 7$
…
$C_{n-1} = 7$
$C_n = 7$

The value of C at any index n is exactly the same as it previous value (index n-1).

So $C_n = C_{n-1}$

Again If we knows C_1, we can fine C_2.
If we knows C_2, we can fine C_3.
…
and so on.

So the initial part is C_1.

Therefore the recursive definition is

$C_1 = 7$
$C_n = C_{n-1}$.

e) $C_n = n^2$

$C_1 = 1^2 = 1$

$C_2 = 2^2 = 4 = C_1 + 3 = C_1 + 2(2)-1$

$C_3 = 3^2 = 9 = C_2 + 5 = C_2 + 2(3)-1$

$C_4 = 4^2 = 16 = C_3 + 7 = C_3 + 2(4)-1$

$C_5 = 5^2 = 25 = C_4 + 9 = C_4 + 2(5)-1$

$C_6 = 6^2 = 36 = C_5 + 11 = C_5 + 2(6)-1$

$C_7 = 7^2 = 49 = C_6 + 13 = C_6 + 2(7)-1$

$C_8 = 8^2 = 64 = C_7 + 15 = C_7 + 2(8)-1$

$C_n = C_{n-1} + 2n - 1$

However that is not the recursive definition because the value of C_n is not only from previous value(s) but also depends on n.

Let look at C_7 that is $C_6 + 13$. Consider the 13. It is from 11 + 2. The 11 is the difference between C_6 and C_5.

So actually, $C_7 = C_6 + 13$

$= C_6 + 11 + 2$

$= C_6 + C_6 – C_5 + 2$

$= 2C_6 - C_5 + 2$

The same as $C_6 = C_5 + 11$

$= C_5 + 9 + 2$

$= C_5 + C_5 – C_4 + 2$

$= 2C_5 - C_4 + 2$

The same as $C_5 = 2C_4 – C_3 + 2 = 2(16) – 9 + 2$

$= 25$

$C_4 = 2C_3 – C_2 + 2 = 2(9) – 4 + 2 = 16$

So the recursive part is $C_n = 2C_{n-1} – C_{n-2} + 2$.

Now if you want to a value of C at index n, you need to know two previous values of index n that are C_{n-1} and C_{n-2}.

So If we know C_1 and C_2 then we can find C_3.
If we know C_2 and C_3 then we can find C_4.
If we know C_3 and C_4 then we can find C_5.
…
and so on.

If we know C_1 and C_2 then we can recursively find any C_n.

Therefore the recursive definition is:

$C_1 = 1$
$C_2 = 4$
$C_n = 2C_{n-1} - C_{n-2} + 2$.

f) $C_n = 2 - (-1)^n$

$C_1 = 2 - (-1)^1 = 2 - (-1) = 2+1 = 3$
$C_2 = 2 - (-1)^2 = 2 - (1) = 2-1 = 1$
$C_3 = 2 - (-1)^3 = 2 - (-1) = 2+1 = 3$
$C_4 = 2 - (-1)^4 = 2 - (1) = 2-1 = 1$
$C_5 = 2 - (-1)^5 = 2 - (-1) = 2+1 = 3$
$C_6 = 2 - (-1)^6 = 2 - (1) = 2-1 = 1$
…

It is almost the same as question 1.d). But its value is from the second previous value.
$C_n = C_{n-2}$

If we know C_1, we can find C_3.
If we know C_3, we can find C_5.
If we know C_5, we can find C_7.
If we know C_7, we can find C_9.
…, and so on.

However only the C's of odd index. If we want to know C_4, we need C_2.
If we know C_2, we can find C_4.
If we know C_4, we can find C_6.
If we know C_6, we can find C_8.
If we know C_8, we can find C_{10}.
…, and so on for all even index.

That means, to find any C, the initial values must be C_1 and C_2.

Therefore the recursive definition is:

$C_1 = 3$
$C_2 = 1$
$C_n = C_{n-2}$.

7. (8 points) For $n \geq 0$ let F_n denote the nth Fibonacci number. Prove that

$$F_0 + F_1 + F_2 + \cdots + F_n = \sum_{1=0}^{n} F_i = F_{n+2} - 1$$

$$F_0 + F_1 + F_2 + \ldots + F_n = \sum_{i=0}^{n} F_i = F_{n+2} - 1.$$

Recursive Definition of Fibonacci number:

$F_0 = 0$
$F_1 = 1$
$F_n = F_{n-1} + F_{n-2}$

Proof by Mathematical Induction

Basic Step: *n = 0 the first valid value of* $\sum_{i=0}^{n} F_i$.

$$s(0) = F_{0+2} - 1 = F_2 - 1$$

from recursive definition $= F_1 + F_0 - 1$

$$= 1 + 0 - 1$$

$$= 0 \qquad (1)$$

$$\sum_{i=0}^{0} F_i = F_0 \qquad (2)$$

(1) = (2). So it is true on at the basic step.

Inductive Step: n = k+1

$$s(k+1) = F_{k+1+2} - 1 = F_{k+3} - 1 \qquad (3)$$

$$\sum_{i=0}^{k+1} F_i = F_0 + F_1 + F_2 + F_3 + \ldots + F_k + F_{k+1}.$$

Since we assume it is true until k, we split the $\sum_{i=0}^{k+1} F_i$ into two parts, $\sum_{i=0}^{k} F_i$ and F_{k+1}.

k+1 k

$$\sum_{i=0} F_i \quad = \quad \sum_{i=0} F_i \quad + \quad F_{k+1}$$

Since we do believe it is true until k, $\sum_{i=0}^{k} F_i = F_{k+2} - 1$

$$\sum_{i=0}^{k+1} F_i \quad = \quad (F_{k+2} - 1) \quad + \quad F_{k+1}$$

$$= \quad F_{k+2} \quad + \quad F_{k+1} \quad - \quad 1$$

By the recursive definition $F_n = F_{n-1} + F_{n-2}$, F_n is from F of its two previous values.

$F_{n+1} = F_n + F_{n-1}$
$F_{n+2} = F_{n+1} + F_n$
$F_{n+3} = F_{n+2} + F_{n+1}$
So $F_{k+3} = F_{k+2} + F_{k+1}$

From $\sum_{i=0}^{k+1} F_i \quad = \quad F_{k+2} \quad + \quad F_{k+1} \quad - \quad 1$

$$= \quad F_{k+3} \quad - \quad 1 \qquad (4)$$

(3) = (4). So it is also true at the inductive step.

It is true for both basic and inductive steps. Therefore

$$F_0 + F_1 + F_2 + \ldots + F_n \quad = \quad \sum_{i=0}^{n} F_i \quad = \quad F_{n+2} - 1.$$

8. (6 points each, total 12 points) Give a recursive definition for the set of all
    a. Positive even integers.
    b. Nonnegative even integers.

a) Recursive definition of the set of all positive even integer.

$C_1 = 2$
$C_2 = 4 = 2+2 = C_1 + 2$

$C_3 = 6 = 4+2 = C_2 + 2$
$C_4 = 8 = 6+2 = C_3 + 2$
…
$C_n = C_{n-1} + 2.$

Again if we know C_1, we can find C_2, then C_3, then C_4, and so on to any C_n.

Therefore the recursive definition is:
$C_1 = 2$
$C_n = C_{n-1} + 2.$

b) Recursive definition of the set of all nonnegative odd integer.

Almost the same as even, the recursive part is $C_n = C_{n-1} + 2$ because it is also the difference of 2. But the initial value, C_1 is 1, (the first value of nonnegative odd integer.)

Therefore the recursive definition is:
$C_1 = 1$
$C_n = C_{n-1} + 2.$

*** However if the question a) is the recursive definition of the set of all nonnegative even integer. C_1 is not 2 but C_1 is 0, the first value, instead.