Administrivia

- (None?)

Binary Relations — Review

- Idea of a binary relation is to express relationship between pairs of elements of a set. Formal definition is in terms of sets of ordered pairs.
- Several properties of interest:
  - $\rho$ is reflexive if $x \rho x$ for all $x \in S$.
  - $\rho$ is symmetric if $(x \rho y) \rightarrow (y \rho x)$ for all $x, y \in S$.
  - $\rho$ is transitive if $(x \rho y) \land (y \rho z) \rightarrow (x \rho z)$ for all $x, y, z \in S$.
  - $\rho$ is antisymmetric if $(x \rho y) \land (y \rho x) \rightarrow (x = y)$ for all $x, y \in S$. 

Partial Ordering

- Idea: Generalize idea of “ordering” to include relations where not all pairs of elements can be ordered.
- Definition: $\rho$ is a partial ordering if it’s reflexive, antisymmetric, and transitive.
- Examples: $\leq$ on integers or reals, $\subseteq$ on sets.
- If finite, can represent with “Hasse diagram” (see examples in textbook).
- Related terms:
  - Successor, predecessor, immediate predecessor, immediate successor.
  - Least, greatest elements.
  - Minimal, maximal elements.

Equivalence Relation

- Idea: Generalize idea of “equals” to include relations where pairs of elements are equivalent but not identical.
- Definition: $\rho$ is an equivalence relation if it’s reflexive, symmetric, and transitive.
- Examples: $=$ on integers or reals, $(x \mod n) = (y \mod n)$ for some $n$.
- Related terms/ideas:
  - Equivalence classes.
  - Partition of a set.
Closures

- We can also talk about the “closure” of a relation with respect to a property — the smallest superset of the relation that has the property.

- Example: Define relation $\rho$ on integers such that $x \rho y$ iff $y = x + 1$. What is the transitive closure of $\rho$?

Uses of Partial Orderings

- One thing a partial ordering (reflexive, symmetric, transitive relation — think “generalized $\leq$”) can express — ordering constraints among tasks.

- We’ll look at one application — topological sorting. PERT charts discussed in book.
Topological Sorting

- Idea here is to take a partial ordering and find a way to extend it to a “total” ordering (i.e., add pairs so that for every $x$ and $y$ either $x \rho x$ or $y \rho x$. How is this useful? e.g., find a way to “schedule” interdependent tasks.
- Notice that there could be more than one way to do this for a given partial ordering.
- How to do this? Next slide … (May not be covered in class.)

Topological Sorting, Continued

- Algorithm for finding a way to extend a partial ordering — “topological sort”:
  - Start with set $S$ and partial ordering $\rho$ on $S$. Idea is to turn $S$ into a sequence $x_1, x_2, \ldots$ such that $(x_i \rho x_j) \rightarrow (i \leq j)$.
  - The algorithm might look like this in pseudocode:
    
    ```
    while (S not empty)
        pick a minimal element $x$ in $S$
        make it the next element of the sequence and remove it from $S$
    end while
    ```

- Does this work? i.e., does it produce an ordering that extends $\rho$? True if we can be sure that for $x$ and $y$ with $x \rho y \rho x$ is picked before $y$. 
Minute Essay

- None — quiz.